Stein-Malliavin method in the Poisson framework

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Limit theorems overview

- **Input alea** (Gaussian process, Poisson process, IID variables,...): X
- **Functional**: $F = F(X)$ (regular., local dep., finite order interactions
  $\rightarrow$ finite expansion, decreasing expansion)
- **Target Law** (Gaussian, Poisson, Gamma, ...): U
- **Tools on the target law**: Stein's method.
- **Tools on the input law**: Stoch. analysis, Malliavin calculus, ...
- **Distance used:** Let $\mathcal{H}$ be some class of functions.

$$d_{\mathcal{H}}(F(X), U) = \sup_{h \in \mathcal{H}} |\mathbb{E}h(F(X)) - h(U)|$$

Distances

The distance used can be

- Kolmogorov distance

$$\begin{aligned} d_K(F,U) &= \sup_{t\in\mathbb{R}} |\mathbb{P}(F \leqslant t) - \mathbb{P}(U \leqslant t)| \\ &= \sup_{t\in\mathbb{R}} |\mathbb{E}h_t(F) - \mathbb{E}h_t(U)| \end{aligned}$$

  i.e. $\mathcal{H} = \{h_t ; t \in \mathbb{R}\}$, where $h_t(x) = 1_{\{x \leqslant t\}}$. More irregular and difficult to handle, more suited to stat. applications.

- Wasserstein distance

$$d_W(F,U) = \sup_{h \text{ 1-Lipschitz}} |\mathbb{E}h(F) - \mathbb{E}h(U)|$$

  more smooth, $\mathcal{H} = 1 -$ Lipschitz functions

Stein's method

- Let N be a standard Gaussian. Then

$$\mathbb{E}Nf(N) - f'(N) = 0$$

for f smooth. If reciprocally some variable V satisfies

$$\mathbb{E}Vf(V) - f'(V) = 0,$$

then V is Gaussian.

- The idea of Stein's method (with a Gaussian target law) is that if $F \approx N$, then

$$\mathbb{E}Ff(F) - f'(F) \approx 0$$

Stein's method

- The trick is to write that for each $h \in \mathcal{H}$, there is f_h such that

$$h(x) - \mathbb{E}h(N) = f_h'(x) - x f_h(x) \qquad \text{(1t order ODE)}$$

so that replacing x with F and taking the expectation yields

$$|\mathbb{E}h(F) - h(N)| \leqslant \sup_h |\mathbb{E}f_h'(F) - F f_h(F)|.$$

The whole point is that we can choose f_h such that $\|f_h\|_\infty, \|f_h\|'_\infty \leqslant c$ and

  - For Wasserstein distance: $, \|f_h''\|_\infty \leqslant c'$
  - For Kolmogorov distance $|f_h''(x)| \leqslant c'' + |x|$.

- We have for some adapted distance

$$d(F, N) \leqslant \sup_f |\mathbb{E}f'(F) - F f(F)| \tag{1}$$

for f in a class depending on the distance.

Another basic idea

The point is therefore to bound

$$|\mathbb{E}f'(F) - Ff(F)|$$

for f with bounded derivatives.

- An idea that works is to look for a variable T such that

$$\mathbb{E}T = 1,$$
$$\text{var}(T) \text{ is small},$$
$$\mathbb{E}Ff(F) \sim \mathbb{E}Tf'(F)$$

  because then

$$\begin{aligned}\left|\mathbb{E}f'(F) - Ff(F)\right| &\leqslant \left|\mathbb{E}f'(F)(T-1)\right| + \mathbb{E}\left|Ff(F) - Tf'(F)\right| \\ &\leqslant c\mathbb{E}\left|T-1\right| + \left|\mathbb{E}Ff(F) - Tf'(F)\right|\end{aligned}$$

How to choose T?

- Malliavin calculus:

$$T = \text{``}\langle DV, -DL^{-1}F\rangle\text{''}$$

  - Gaussian framework: $\mathbb{E}Tf'(F) - Ff(F) = 0$!
  - Poisson framework: Need some kind of chain rule to bound $|\mathbb{E}Tf'(F) - Ff(F)|$
- Zero-bias transform: Build F^* such that for f smooth

$$\mathbb{E}Ff(F) = \mathbb{E}f'(F^*).$$

- Chatterjee (binomial framework). Explicit construction of a variable T such that

$$\mathbb{E}f'(F^*) \approx \mathbb{E}[f'(F)\mathbb{E}(T|F)],$$

i.e. in some sense $\mathbb{E}(T|F) \approx \frac{dF^*}{dF}$, and therefore

$$\mathbb{E}[Ff(F) - Tf'(F)] \text{ is small}$$

Poisson random measures

- E underlying Polish space (typically $E \subset \mathbb{R}^d$)
- μ Radon measure on E
- X Poisson random measure with intensity μ; noted $X \sim \mathcal{P}(\mu)$,such that
  - If $A, B \subset E, A \cap B = \emptyset$, then $X(A), X(B)$ are independent
  - $X(A)$ is Poisson with parameter $\mu(A)$
- Since a Poisson measure with parameter n is much concentrated in $[n/2, 2n]$, X is quite close to n IID points $X_1, \ldots, X_n$ with $n = \mu(E)$.
- If μ has no atoms, then a.s. X is a finite sum of Diracs in distinct points, and we consider X as a locally finite set $X = \{x_n; n \geqslant 1\}$
- We introduce the compensated Poisson measure $\overline{X} = X - \mu$.

For simplicity we assume that μ has no atoms and X is a set

Stochastic integrals

For $f \in L^2(E^k; \mu^k)$ symmetric vanishing on the diagonal, one can define the centred random variable

$$I_k(f_k) = \int f d\overline{X}^k.$$

For f_k non-vanishing on the diagonal, we have

$$I_k(f_k) := I_k(\overline{f_k})$$

where $\overline{f_k}$ is zero on the diagonal and equal to f_k elsewhere.
The k-th Gaussian chaos is the class of all such integrals, and they satisfy, for $m, k \geqslant 0$, $f \in L^2_s(E^k), g \in L^2_s(E^m)$

$$\mathbb{E} I_k(f) I_m(g) = k! 1_{m=k} \langle f, g \rangle$$

U-statistics

- More likely to appear in applications are the Poisson U-statistics:

$$U_k(f;X) = \int f dX^k = \sum_{(x_1,\dots,x_k)\in X_k^{\neq}} f(x_1,\dots,x_k)$$

  whenever this is well defined.
- It is easy to pass from U-statistics to stochastic integrals and vice versa (assume $f=0$ on the diagonal):

$$\int f dX^k = \int f d((X-\mu)+\mu)^k = \sum_{n=0}^{k} \binom{k}{n} \int f_n d\overline{X}^k = \sum_{n=0}^{k} \binom{k}{n} I_n(f_n)$$

  where

$$f_n(x_1,\dots,x_n) = \int f(x_1,\dots,x_n,x_{n+1},\dots,x_k) dx_1 \dots dx_n.$$

  more convenient notation:

$$f_n(\mathbf{x}_n) = \int f(\mathbf{x}_n,\mathbf{x}_{k-n}) d\mu^{k-n}(\mathbf{x}_{k-n})$$

Wiener-Ito decomposition

Every L^2-variable $F(X)$ admits an infinite decomposition

$$F(X) = \sum_{n \geqslant 0} I_n(f_n)$$

It will turn out that the kernels $f_n \in L^2(E^n)$ can actually be expressed in terms of the n-th order Malliavin derivatives of F.
Proof:

- $G = \{\mu \mapsto e^{-\mu(h_1)} + \cdots + e^{-\mu(h_k)}\}$
- $\{I_k(h)\}$ dense in G
- $\mu(C) = \lim_{t \to 0} \frac{1}{t}(1 - e^{-t\mu(C)})$

Notation

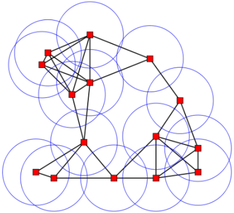
- Let $\lambda > 0$
- ℓ : Lebesgue measure
- $X = X_\lambda = \{x_i; i \geqslant 1\}$: a homogeneous Poisson process with intensity some loc.finite measure μ_λ.
- $X_k^{\neq}$: Set of k-tuples of distinct points of X.

Subgraph counting (regimes classification: Penrose 2003)

- Let $\alpha_\lambda > 0$,
- Let $\mathcal{G} = (X, \mathcal{E})$ be the graph with edges $\mathcal{E} = \{(x_i, x_j) : \|x_i - x_j\| \leqslant \alpha_\lambda\}$ (Gilbert graph).
- Let G be a finite connected graph (e.g. G is a triangle)

Then the following random variable is a Poisson U-statistic of order $\#G$:

$$U_G(X) = \#\{\text{occurrences of } G \text{ as a subgraph of } \mathcal{G}\}?$$



- $\mu_\lambda = \ell 1_{\lambda^{1/d}[0,1]^d}$, $k = \#G$
- $f(x_1, \ldots, x_k) = 1_{\{\text{the subgraph induced on the } x_i' s \text{ contains } G\}}$.

U-statistics in the boolean model

- ν: Probability measure on the class of compact sets $\mathcal{K}$.
- To each x_i (the germs) is attached an independent random compact set K_i (the grain) with distribution $\nu(dK_i)$.
- Put $X' = \{(x_i, K_i)\}$, a marked Poisson point process (product ctrl measure).
- Two points x_i, x_j are *connected* if $(x_i + K_i) \cap (x_j + K_j) \neq \emptyset$.

Example: Number of intersections in a process of line segments.

- ν charges isotropically segments centered in 0.
- Number of segment intersections.

$$U_\lambda := \sum_{i \neq j} 1_{\{x_i \text{ is connected to } x_j\}}$$

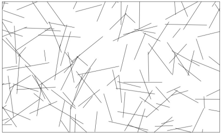


Figure: Zbynek Pawlas

Sylvester's problem

Given a convex body $W \subset \mathbb{R}^d$, $k \geqslant d+1$, and $X_1, \ldots, X_k$ IID uniform in W, what is

$$p_k(W) = \mathbb{P}(X_1, \ldots, X_k \text{ are in convex position})?$$

- Explicit formulae for the square, the triangle, or other basic shapes exist. In general, how to estimate $p_k(W)$?
- If k is large, the accept-reject method is unrealistic.

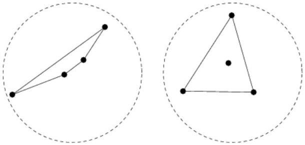


Figure: Are 4 random points in convex position?

- Reitzner & Schulte 2013 proposed the unbiased estimator

$$\hat{p}_{k,\lambda}(W) = (\lambda\ell(W))^{-k} \sum_{\mathbf{x}_k := (x_1,\dots,x_k)\in X_k^{\neq}} \mathbb{1}_{\{x_1,\dots,x_k \text{ are in convex position}\}}$$

- The problem is equivalent if the control measure is

$$\ell 1_{\lambda^{1/d} W}.$$

Flat processes

- Let Γ be a locally finite measure on the set of lines of $\mathbb{R}^d$ invariant under translations.
- Let $[W]$ be the class of lines that intersect $W \subset \mathbb{R}^d$ a convex body.
- Let $X = \{x_i : i \geqslant 1\}$ be a Poisson process with control measure $\lambda\Gamma 1_{[W]}$.
- What are the properties of

$$\#(\{x_i \cap x_j \cap W; i < j\})?$$

- Same question with a process of m-dimensional affine subspaces of $\mathbb{R}^d$, with $m > d/2$.
- equivalent problem with control measure $\ell 1_{\lambda^{1/d}[W]}$

Stationarity and scaling

In our applications, the kernel h_λ satisfies

$$h_\lambda(\mathbf{x}_k) = h_0(\alpha_\lambda \mathbf{x}_k)$$

for some scaling factor $\alpha_\lambda > 0$, where h_0 is invariant under translations and permutations:

$$h_0(x_1 + y, \dots, x_k + y) = h_0(x_1, \dots, x_k),\ y \in \mathbb{R}^d$$

and

$$\mathbf{x}_{k-1} \mapsto h_0(0, \mathbf{x}_{k-1})$$

is "small" far from the origin

Malliavin derivative

Technical assumptions all along:

- Every variable has to be L^2
- Every kernel has to be L^2 and symmetric

Small perturbation of X in some point $x \in E$: $X \to X \cup \{x\}$.

$$D_x F(X) = F(X \cup \{x\}) - F(X).$$

Under the expansion representation,

$$D_x \left(\sum_{n \geqslant 0} I_n(f_n) \right) = \sum_{n \geqslant 1} n I_n(f_{n-1}(x, \cdot))$$

We also have for ε small, ν a (finite sum of)Dirac measure(s),

$$F(X + \varepsilon \nu) \approx F(X) + \varepsilon \langle DF, \nu \rangle$$

Integration by parts

For u under the representation

$$u_x = \sum_{n\geqslant 0} I_n(f_n(x,\cdot))$$

$$\delta(u_x) = \int \left(\sum_{n\geqslant 0} \int f_n d(X-\mu)^n \right) d\bar{X}$$

$$= \sum_{n\geqslant 0} \int f_n(x,\dots) d(X-\mu)^n \otimes (X-\mu)$$

Under some assumptions on u, we can write

$$\delta(u) := \text{“}I_1(u) =\text{''} \sum_{x\in X} u_x - \int u_x d\mu(x).$$

The Malliavin derivative is such that

$$\mathbb{E} F\delta(u) = \mathbb{E}\langle DF, u\rangle.$$

Higher order derivatives

$$D^2_{x,y}F = D_x D_y F = D_y D_x F$$
$$D^n_{x_1,\dots,x_n}F = D_{x_1}\left(D^{n-1}_{x_2,\dots,x_{n-1}}F\right)$$

We have for $F \in L^2$

$$f_n(x_1,\dots,x_n) = \frac{1}{n!}\mathbb{E}D_{x_1,\dots,x_n}F.$$

measures the n-th order interactions between the points. (used for Poisson Voronoi approximation)
For instance,

$$\begin{aligned} f_2(x,y) &= \frac{1}{2}\mathbb{E}\left(D_x(D_yF)\right) \\ &= \frac{1}{2}\mathbb{E}\left(F(X\cup\{x,y\} - F(X\cup\{y\}) - (F(X\cup\{x\}) - F(X))\right) \end{aligned}$$

Orstein-Uhlenbeck operator

We introduce

$$LF = -\delta DF$$

Under the expansion representation

$$LF = -\sum_{n \geqslant 0} n I_n(f_n)$$

and its inverse on the space of centred functionals

$$L^{-1}F = -\sum_{n \geqslant 0} \frac{1}{n} I_n(f_n)$$

Stein-Malliavin bound (Peccati, Solé, Utzet, Taqqu 2010)

It yields for F centred with variance 1

$$\begin{aligned}\mathbb{E}Ff(F) = \mathbb{E}(LL^{-1}F)f(F) &= -\mathbb{E}\delta D(L^{-1}F)f(F) = -\mathbb{E}\langle DL^{-1}F, Df(F)\rangle \\ &= -\mathbb{E}\langle DL^{-1}F, f'(F)DF + R(DF)\rangle\end{aligned}$$

where $|R(DF)| \leqslant |DF|^2$ because $\|f''\|_\infty \leqslant 2$.
Then, with $T = \langle DL^{-1}F, -DF\rangle$

$$\begin{aligned}\left|\mathbb{E}Ff(F) - f'(F)\right| &\leqslant \left|\mathbb{E}(f'(F))(T-1)\right| + \int_E \mathbb{E}\left|(D_xF)^2 D_x L^{-1}F\right| \mu(dx) \\ &\leqslant c\sqrt{\mathrm{var}(T)} + \int_E \mathbb{E}\left|(D_xF)^2 D_x L^{-1}F\right| \mu(dx)\end{aligned}$$

($\mathbb{E}T = \mathrm{var}(F) = 1$)

Product formula and contractions

- In the formula above appear products of the form $I_m(f)I_k(g)$ and we can prove that under technical assumptions

$$I_m(f)I_k(g) = \sum_{0\leqslant l\leqslant r\leqslant m\wedge k} \varkappa_{r,l} I_{m+k-r+l}(f \star_r^l g)$$

where $f \star_r^l g$ is a "contraction" and has an explicit integral expression:

$$f \star_r^l g(\mathbf{x}_{m-r}, \mathbf{x}'_{k-r}, \mathbf{y}_{r-l}) = \int f(\mathbf{x}_{m-r}, \mathbf{y}_{r-l}, \mathbf{z}_l) g(\mathbf{x}'_{k-r}\mathbf{y}_{r-l}, \mathbf{z}_l)\mu^l(\mathbf{z}_l)$$

- Then if $F = U_k(h) = \sum_{1\leqslant m\leqslant k} I_m(h_m)$,

$$d_W(\tilde{F}, N) \leqslant C_k \underbrace{\left(\max_{1\leqslant l\leqslant r\leqslant k, l\neq k} \| h_m \star_r^l h_n \|_2 + \max_{m=1}^{k} \| h_m \|_{L^4}^2 \right)}_{M}$$

Application: 4th moment theorem (LR-Peccati 1, p.14)

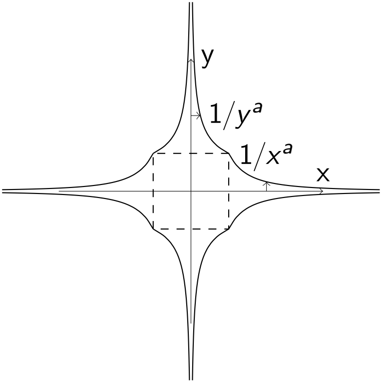
Assume $F = \sum_{k=0}^{n} I_k(f_k)$ with each $f_k \geqslant 0$, $\mathbb{E}F = 0, \mathbb{E}F^2 = 1$. Then

$$cM\| \leqslant \sqrt{\mathbb{E}F^4 - 3} \leqslant CM$$

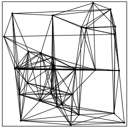
Therefore if F^4 is uniformly integrable, it converges to N iff its 4 first moments do.
example: Counting functionals

Different regimes: $H_\lambda = \alpha_\lambda H_1$

H_1:

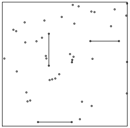


$a = 1, \alpha_\lambda = 1, \lambda = 25$:



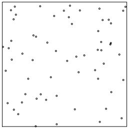
CLT at speed $\lambda^{-1/2}$

$a = 1, \alpha_\lambda = \lambda^{-1/d}, \lambda = 50$:



CLT in $\log(\lambda)^{-1}$

$a = 2, \alpha_\lambda = \lambda^{-1/d}, \lambda = 50$:



no CLT.

Interaction volume

We come back to the U-statistics examples. They are under the form

$$U_k(h) = \sum_{(x_1,\dots,x_k)\in X_k^{\neq}} h(x_1,\dots,x_k) = \sum_{(x_1,\dots,x_k)\in X_k^{\neq}} h_0(\alpha_\lambda(x_1,\dots,x_k))$$

call $v_\lambda = \alpha_\lambda^d$ the *interaction volume*. Since $h_\lambda(\mathbf{x}_k) = h_0(\alpha_\lambda \mathbf{x}_k)$ and h_0 is small far from the diagonal, v_λ is the magnitude of the number of points having a significant influence on the contribution of a typical point.

- Subgraph counting: $v_\lambda \sim \alpha_\lambda^d$
- Boolean model: $v_\lambda \sim 1$ (integrability condition on the random radii)
- Sylvester problem, flat processes: $v_\lambda \sim \lambda$

Assumptions

- X_λ Poisson process with measure $\ell 1_{\lambda^{1/d}W}$
- Scaled kernel: $h_\lambda(\mathbf{x}_k) = h_0(\alpha_\lambda \mathbf{x}_k)$
- h_0 is stationary and rapidly decreasing away from the diagonal: there is a probability density $0 < \varkappa \leqslant C < \infty$ such that for $p = 2, 4$,

$$\int h_0(0, \mathbf{x}_{k-1})^p \varkappa(x_{k-1})^{1-p} dx_k < \infty.$$

- h_0 has non-zero projections: for $1 \leqslant m \leqslant k$

$$\mathbf{x}_m \mapsto \int h_0(\mathbf{x}_m, \mathbf{x}_{k-m}) d\mathbf{x}_{k-m} \not\equiv 0,$$

Main result (LR-Peccati)

Then we have

$$0 < C_1 \leqslant \frac{\mathrm{var}(U_k(h_\lambda))}{\lambda v_\lambda^{2k-2} \max(1, v_\lambda^{-k+1})} \leqslant C_2 < \infty$$

and calling

$$\tilde{U}_k(h_\lambda) = \frac{U_k(h_\lambda) - \mathbb{E} U_k(h_\lambda)}{\sqrt{\mathrm{var}(U_k(h_\lambda))}}$$

we have

$$d(\tilde{U}_k(h_\lambda), N) \leqslant C_3 \lambda^{-1/2} \max(1, v_\lambda^{-(k-1)/2})$$

where d is either

- d_W the Wasserstein distance
- d_K the Kolmogorov distance (using Thaële & Eischelsbacher 2013)
- and N is a standard Gaussian variable

Regimes ($c > 0$ constant)

CLT Regimes:

- $v_\lambda \gg c$: CLT with distance $\leqslant \lambda^{-1/2}$. Low order chaoses dominate.
- $v_\lambda \sim c$: CLT with distance $\leqslant \lambda^{-1/2}$. Chaoses co-dominate.
- $\lambda^{-1/(k-1)} \ll v_\lambda \ll c$: CLT with distance $\ll^{-1/2}$. High order chaoses dominate.

Applies to subgraph counting, flat processes, Sylvester problem, and segment intersections if

$$\mathbb{P}(\text{length}(K_1) \geqslant r) \leqslant Cr^{-5-\varepsilon}, r \geqslant 1.$$

Non-CLT Regimes:

- $v_\lambda \sim \lambda^{-1/(k-1)}$: Poisson limit.
- $v_\lambda \ll \lambda^{-1/(k-1)}$: Zero.

Multidimensionnal CLTs (see p.18-19 of Peccati and Zheng)

Let $F_n = (F_1, \ldots, F_q)$ where the F_i's are Poisson functionals. Let $N \sim \mathcal{N}(0, C)$ where C is a m-dimensional covariance matrix.
We have

$$d_{\mathcal{H}}(F, N) \leqslant c \left(\sqrt{\sum_{i,j=1}^{q} [C_{i,j} - \mathbb{E}\langle DF_i, -DL^{-1}F_j \rangle]} \right.$$
$$\left. + \int \left(\sum_{i=1}^{q} |D_x F_i|^2 \right) \left(\sum_{i=1}^{q} |D_x L^{-1} F_i| \right) \mu(dx) \right)$$

where $h \in \mathcal{H}$ if $h \in \mathcal{C}^3$ with $\|h''\|_\infty, \|h'''\|_\infty \leqslant 1$.

Poisson limits

- Let $c > 0$

Theorem (Peccati 2012)

If $F \in L^2$ has integer values, then

$$d_{TV}(F, \mathcal{P}(c)) \leqslant \frac{1-e^{-c}}{c}\mathbb{E}\left|c - \langle DF, -DL^{-1}F\rangle\right| + \frac{1-e^{-c}}{c^2}\mathbb{E}\left(\int_E \left|D_xF(D_xF-1)D_xL^{-1}F\right|\right)$$

Stein's equation for the Poisson law: If F has integer values, it is $\mathcal{P}(c)$ iff it satisfies

$$\mathbb{E}cf(X+1) - Xf(X) = 0$$

- If F does not have integer values, one introduces perturbations B such that $F - B \in \mathbb{Z}$.

Poisson limit (II)

If $F_n = I_q(f_{q,n}) + B_n$ with $q \geqslant 2$ and $\mathbb{E}F_n \to c, \mathrm{var}(F_n) \to c, B_n \to 0$. If for $1 \leqslant l \leqslant r \leqslant q, l \neq q$,

$$\|f_{n,q} \star_r^l f_{n,q}\|_{L^2} \to 0$$

and

$$\int \left(f_n^2 + q!^2 f_n^4 - 2q! f_n^3\right) d\mu^q \to 0,$$

then $d_{TV}(F_n, \mathcal{P}(c)) \to 0$.

Portmanteau-inequality (Bourguin-Peccati 2012)

- $X \sim \mathcal{P}(c)$
- $N \sim \mathcal{N}(0,1)$
- F, G Poisson functionals, $\mathbb{E}F = c, F \in \mathbb{Z}_+, \mathbb{E}G = 0, \mathrm{var}(G) = 1$

$$d_{\mathcal{H}}((F,G),(X,N)) \leqslant$$

$$C\left(\mathbb{E}\left|c - \langle DF, -DL^{-1}F\rangle\right| + \mathbb{E}\left|1 - \langle DG, -DL^{-1}G\rangle\right|\right.$$

$$+\mathbb{E}\int \left|D_xF(D_xF-1)D_xL^{-1}F\right|\mu(dx)$$

$$+\mathbb{E}\int \left|D_xG\right|^2\left|D_xL^{-1}G\right|\mu(dx)$$

$$\left.+\mathbb{E}\langle\left|DL^{-1}G\right|, |DF|\rangle\right)$$

where $h(x,y) \in \mathcal{H}$ if $|h| \leqslant 1$ and h is 1-Lipschitz in y.

Semi-group representation and Glauber dynamics

Consider the following dynamics on the Poisson measure $X_t, t > 0$:

- $X_0 = X$
- Each point of the Poisson process disappears independently with rate 1.
- New points appear according to the rate $\mu(dx)dt$.
- Locally $X_t \to X_\infty$ an independent Poisson measure as $t \to \infty$.
- Given a functional f, we define the semigroup

$$P_t f(x) = \mathbb{E}' f(X_t | X_0 = x)$$

for any locally finite point configuration x.

Mehler's formula

(Privault, ...)

- It turns out that for $t > 0$, with $T_t = P_{e^{-t}}$,

$$P_t\left(\sum_{n\geqslant 0} I_n(f_n)\right) = \sum_{n\geqslant 0} e^{-nt} I_n(f_n)$$

- We have for $F \in L^2$ centred (take $F = I_q(f)$)

$$LF = -qI_q(f) = \lim_{t\to 0} \frac{e^{-qt}-1}{t} F = \lim_{t\to 0} \frac{P_tF - F}{t}$$

$$L^{-1}F = -\int_{t>0} P_tFdt$$

Remember that $L^{-1}F = -n^{-1}F$. And

$$P_tF = e^{-nt}F$$

and therefore

$$\int_{t>0} P_tFdt = -L^{-1}F.$$

Last-Peccati-Schulte 2014

- Second order Poincaré inequality
- $$d_W(F,N) \leqslant \gamma_1 + \gamma_2 + \gamma_3$$
  $$d_K(F,N) \leqslant \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 + \gamma_5 + \gamma_6$$
- General inequalities and stabilization

Malliavin derivative over binomial input?

Let $X = X_1, \ldots, X_n$ be IID variables, $X' = (X_1', \ldots, X_n')$ independent copies, and $F = F(X_1, \ldots, X_n)$ a L^2 functional. Define

$$D_i F(X, X') = F(X) - F(X^i)$$

where

$$X^i = (X_1, \ldots, X_i', \ldots, X_n).$$

Then F has the orthogonal (Hoeffding) decomposition

$$F = \mathbb{E}F(X) + \sum_{k=1}^{n} \underbrace{\sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} (-1)^k \mathbb{E}\Delta_{i_1} \ldots \Delta_{i_k} F(X)}_{\text{“}I_k(f_k)\text{''}}$$

$$= \mathbb{E}F(X) + \sum_{k=1}^{n} \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} f_{i_1,\ldots,i_k}(X_{i_1}, \ldots, X_{i_k})$$

For $\{i_1, \ldots, i_k\} \neq \{j_1, \ldots, j_m\}$,

$$\mathbb{E}(\Delta_{i_1} \ldots \Delta_{i_k} F)(\Delta_{j_1} \ldots \Delta_{j_m} F) = 0.$$

Application: variance lower bound (symmetric case)

$$\begin{aligned}\mathrm{var}(F) &\geqslant \sum_{i=1}^{n} \mathbb{E}[(\mathbb{E}[D_i F(X', X)|X])^2] \\ &\geqslant n \int (\mathbb{E}[f(X) - f(x, X_2, \ldots, X_n)])^2 \mu(dx)\end{aligned}$$

to be compared with Stein-Efron inequality

$$\mathrm{var}(f(X)) \leqslant \frac{n}{2} \int \mathbb{E}[f(X) - f(x, X_2, \ldots, X_n)]^2$$

The lower bound seems to be sharp when the problem is "inhomogeneous" (e.g. Voronoi set approximation)

- https://sites.google.com/site/malliavinstein/home
- Survey book coming soon (?)