Ensembles convexes et mosaïques aléatoires

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1 Convex hulls of stochastic processes

2 Convex rearrangements

3 Random tessellations

4 Generalised peeling
1. Convex hulls of stochastic processes

2. Convex rearrangements

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4. Generalised peeling
Extremal points

Stochastic process $X$
$\text{ext}(X) : \text{Set of the extremal points of its graph.}$

Topics of interest:

- Topology of $\text{ext}(X)$,
- 1-dimensional Hausdorff measure of $\text{ext}(X)$.

More precisely:
$\text{ext}^+(X) : \text{Extremal points in the upper part of the convex hull.}$
Continuous model

- $I$: Interval of $\mathbb{R}$.
- 1-dimensional fluid uniformly spread along $I$.
- $v(x, t)$: Velocity field of the fluid at time $t$.
- $\psi(x, t)$: Potential, defined by $\partial_x \psi(x, t) = v(x, t)$.

Evolution ruled by Burgers equation (inelastic shocks between particles).
• Initial data: \( X(x) = \psi(x, 0) \), random process.

• \( \mathcal{L}_t \): Set of Lagrangian regular points, i.e. the points not involved in a shock up to time \( t \).

Hopf (1950), Cole (1951): Explicit solution in terms of \( \psi(x, t) \).

Call \( f(x) = -x^2/2 \). \( \mathcal{L}_t \) is the set of points of \( \text{ext}^+(X + t^{-1}f) \) that are isolated neither on the left nor on the right.

Theorem

\[
\text{ext}^+(X + t^{-1}f) \to \text{ext}^+(X) \text{ in the Hausdorff metric as } t \to \infty.
\]

Thus the set \( \mathcal{L}_t \) tends to “shrink” around the points of \( \text{ext}^+(X) \). Explicit description of \( \mathcal{L}_t \): Bertoin, Winkel, Giraud,...
Results of negligibility for $\text{ext}^+(X)$

**Theorem**

Assume we are in one of the following situations:

- $X$ is a reversible “nice” Markov process, meaning for all $s > 0$,

  $$\{X(s + t) - X(s) : 0 \leq t < s\} \overset{(d)}{=} \{X(s) - X((s - t)^-) : 0 \leq t < s\}.$$

  (It includes Lévy processes without drift)

- $X$ is an Ito process which Brownian component vanishes on no open interval.

Then $\text{ext}(X)$ has zero 1-dimensional Hausdorff measure.
Topological results

Variation of $X$ on an interval $I$:

$$V(X, I) = \sup_{x_i, q} \sum_{i=1}^{q} |X(x_i) - X(x_{i+1})|.$$ 

for $0 < x_1 < \cdots < x_q \in I$.

If $X$ Lévy process,

- either $V(X, I) = \infty$ for all open $I$,
- or $V(X, I) < \infty$ for all bounded $I$.

A Lévy process $X$ with bounded variation only varies by jump,

$$X(a) - X(0) - ba = \sum_{0 \leq s \leq a} (X(s^+) - X(s^-)),$$

$b$: Drift, irrelevant here. We assume $b = 0$. 

$T_{\text{max}}$: Time where $X$ approaches its maximal value if $I$ is a compact interval (otherwise, $T_{\text{max}} = +\infty$).

**Theorem**

- $T_{\text{max}} \in \text{ext}^+(X)$,
- $\text{ext}^+(X)$ is discrete away from $T_{\text{max}}$,
- $T_{\text{max}}$ is a left (resp. right) accumulation point of $\text{ext}^+(X)$ iff $X$ takes strictly positive (resp. negative) values arbitrary close from 0.
Concave drift

The set $\mathcal{L}_t$ is determined by $\text{ext}^+(X + t^{-1}f)$, where $f(x) = -x^2/2$.

**Theorem**

$X$: Levy process without drift and with bounded variation.

$f$: Concave function.

$$\text{ext}^+(X - f) \subset \text{ext}^+(X) \subset \text{ext}^+(X + f).$$

Call $\overline{g}$ the concave majorant of a function $g$.

A left (resp. right) accumulation point $s$ in $\text{ext}^+(X + f)$ satisfies

$$\overline{X + f'}(s^-) = f'(s),$$

resp. $$\overline{X + f'}(s^+) = f'(s).$$

If $f = 0$, you retrieve the fact that every accumulation point $s$ of $\text{ext}^+(X)$ satisfies

$$\overline{X'}(s) = 0,$$

whence $X(s)$ is an extremum of $X$. 
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Associated convex body of a 1-dimensional function

Resource distributed to a population of size $N$.

- Member labelled $k$ receives $r_k$.
- Cumulative income function: $f(n) = \sum_{k \leq n} r_k$.

$f$ is extended to a piece-wise linear function on $[0, N]$. 
The area of the convex body can measure the inequalities over this particular resource (consider the equality case, where $r_k$ is equal for all $k$).
Lower part (red): \textit{convex rearrangement of } \( f \), denoted by \( \mathcal{C}f \).
Rearrangement of the derivative

It corresponds to rearranging the derivative in a monotone way. If $f'$ is the derivative of $f$, and $(\mathcal{C}f)'$ the derivative of $\mathcal{C}f$, we have

$$\lambda_1 f'^{-1} = \lambda_1 (\mathcal{C}f)'^{-1}.$$
Asymptotic rearrangement of the Brownian motion
Theorem (Davydov, Zitikis 2004)

$X$: Brownian motion.

$X_n$: Piece-wise linear interpolation of $X$ on $\{0, 1/n, \ldots, 1\}$.

Then

$$\sup_{x \in [0,1]} \left| \frac{1}{\sqrt{n}} cX_n(x) - L(x) \right| \to 0,$$

$L$: Lorenz curve.

Other asymptotic convex rearrangements in Davydov & Vershik 1998.
Multidimensional rearrangement

Let \( f : [0, 1]^d \rightarrow \mathbb{R} \), differentiable a.e. such that

\[
\int_{[0,1]^d} \| \nabla f(x) \| \, dx < +\infty.
\]

A convex function \( C \) is a convex rearrangement of \( f \) if

\[
\lambda_d \nabla f^{-1} = \lambda_d \nabla C^{-1}.
\]

**Theorem (Brenier, 91)**

Every function \( f \) with finite gradient mass has a convex rearrangement. The convex rearrangement is unique up to a constant.
Asymptotic rearrangement

- \( f \): “irregular function"
- \( f_n \): Functions with finite gradient mass, the \( f_n \) converge to \( f \). Is there a function \( C \), and positive numbers \( \{b_n; n \geq 1\} \), such that

\[
\lim_{n \to \infty} b_n C f_n \to C?
\]

If yes, \( C \) is a asymptotic convex rearrangement.

**Theorem**

\( \{f_n; n \geq 1\} \): Functions with finite gradient mass,
\( \{b_n; n \geq 1\} \): Positive numbers.

The following assertions are equivalent

(i) Weak convergence \( \lambda_d \nabla (b_n f_n)^{-1} \Rightarrow \mu \).

(ii) \( b_n C f_n(z) \to C(z) \), for \( z \in \text{int}([0, 1]^d) \),

(iii) \( \nabla (b_n C f_n)^{-1} \to \nabla C \) in the \( L^1 \) sense on every sub-compact, whence \( C \in \mathcal{C} f \).

In this case: \( \mu = \lambda_d \nabla C^{-1} \).
Gaussian fields

$X_n$: Approximations of a Gaussian field $X$ on $[0, 1]^d$. $X_n$ is obtained by interpolation of $X$ on a triangulation $\mathcal{T}_n$. There are regular simplices $T_1, \ldots, T_k$, and a discrete group $\Gamma$ of $\mathbb{R}^d$ such that

$$\mathcal{T}_n = \left\{ \frac{1}{n}(\gamma + T_j) : \gamma \in \Gamma, 1 \leq j \leq k \right\}.$$

Call $\mathcal{V}_n$ the set of vertices of $\mathcal{T}_n$. 
Brownian sheet approximation
Results

Theorem

Assume the following:
Convergence of the mean: There is a deterministic measure $\mu$ such that, for all Borel set $B$,

$$
\mathbb{E} \int_{[0,1]^d} \mathbb{1}_{\{b_n \nabla X_n(z) \in B\}} \, dz = \mathbb{E}(\lambda_d (b_n \nabla X_n)^{-1}(B)) \to \mu(B).
$$

$\Theta$: Set of points $(x, y)$ where the covariance of $X$ is not of class $C^1$

- $\sum_n b_n^{-4} < +\infty$,
- $\sum_n n^{-2d} \text{card}(\Theta \cap V_n^2) < +\infty$,
- $\sup_{n \in \mathbb{N}, z \in [0,1]^d} \text{cov}(\nabla b_n X_n(z)) < +\infty$.

Then

$$b_n cX_n(z) \to C(z), \ z \in \text{int}([0,1]^d)$$

where $C$ is convex and the gradient distribution of $C$ is $\mu$. 
Asymptotic rearrangement of the Brownian sheet
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Construction on a bounded convex window
Wait an exponential time with parameter $\text{Per}(W)$. Then draw a random line.
Construction on a bounded convex window $W$. Each cell created behaves independently. On each cell, wait an exponential time with parameter the perimeter, and then draw a random line inside the cell.
Construction on a bounded convex window $W$. Each cell created behaves independently. On each cell, wait an exponential time with parameter the perimeter, and then draw a random line inside the cell.
Stop the construction at some predetermined time $a > 0$
Properties of the construction:

- The behaviour of two disjoint cells is independent.
- The process has no memory.

The parameters of the construction are

- The time parameter \( a > 0 \),
- The law of the random lines.

The same construction can take place in \( \mathbb{R}^d \), but lines are replaced by hyperplanes.

It can model the cracking of a drying soil, or the subdivision procedure of cells.
Theorem

There exists a tessellation $T$ on all $\mathbb{R}^d$ such that the law of $T \cap W$, for any compact $W$, corresponds to the above tessellation.

Such a tessellation is called STIT tessellation.
Mixing property

For two compact sets $A$ and $B$, and a STIT tessellation $T$, one has

$$
P(A \cap T = \emptyset, (B + h) \cap T = \emptyset) \to P(A \cap T = \emptyset)P(B \cap T = \emptyset),$$

when $\|h\| \to \infty$.

It implies that $T$ is ergodic. Moreover, one has the decay rate

$$|P(A \cap T = \emptyset, (B + h) \cap T = \emptyset) - P(A \cap T = \emptyset)P(B \cap T = \emptyset)| = o(\|h\|^{-1}).$$
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Convex peeling

Convex peeling of a set of points.

For a random set of points $X$, call $X_{(k)}$ the points that are on the border of the $k$-th envelope. We have

$$X = \bigcup_k X_{(k)},$$

and the union is disjoint.

We set

$$\text{env}^k(X) = \text{conv}(X_{(k)}).$$
Weak convergence for $\alpha$-stable point processes

IID random variables: $\xi_i, i \geq 1$, which common law belongs to the attraction domain of an $\alpha$-stable law.

$$X_n = \{ b_n \xi_1, \ldots, b_n \xi_n \}$$

with proper $b_n$,

$$X_n \rightarrow X,$$

$X$: $\alpha$-stable Poisson point process.

Weak convergence:

$$\text{env}^k(X_n) \Rightarrow \text{env}^k(X),$$

The topology is that generated by the Hausdorff metric.

Davydov & Nagaev 2004 deduced the convergence of functionals based on $X(k)$ (extreme summands).
The $\mathcal{F}$-convexity

Closed convex hull

\[
\text{conv}(A) = \bigcap_{A \subseteq H} H,
\]

for $H$ closed half-space.

Closed half-spaces $\Rightarrow$ arbitrary family of level sets, of the form

\[
H = \{ x : f(x) \leq \alpha \},
\]

for $\alpha \in \mathbb{R}$, $f$ continuous function, $f \in \mathcal{F}$.

Convex hull: $\mathcal{F}$ is the class of linear forms.
Convergence result

Define the $\mathcal{F}$-peeling similarly. We obtained sufficient conditions for having the same result of convergence. There is the weak convergence

$$\text{env}_k^F(X_n) \Rightarrow C_k^F,$$

$$C_k^F = \text{env}_k^F(X),$$

for some $\alpha$-stable Poisson point process $X$. 