Ensembles convexes et mosaïques aléatoires

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1. Convex hulls of stochastic processes
2. Convex rearrangements
3. Random tessellations
4. Generalised peeling

1. Convex hulls of stochastic processes

2. Convex rearrangements

3. Random tessellations

4. Generalised peeling

Extremal points

Stochastic process X

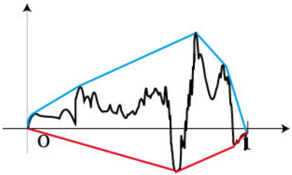
$\mathrm{ext}(X)$: Set of the extremal points of its graph.

Topics of interest:

- Topology of $\mathrm{ext}(X)$,
- 1-dimensional Hausdorff measure of $\mathrm{ext}(X)$.

More precisely:

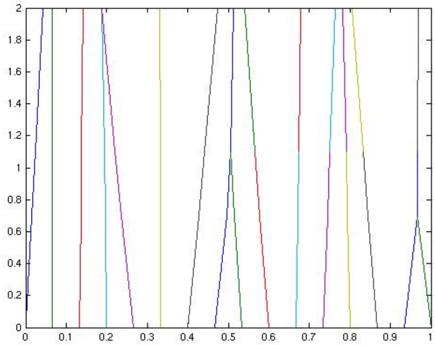
$\mathrm{ext}^+(X)$: Extremal points in the upper part of the convex hull.



Continuous model

- I: Interval of $\mathbb{R}$.
- 1-dimensional fluid uniformly spread along I.
- $v(x, t)$:Velocity field of the fluid at time t.
- $\psi(x, t)$: Potential, defined by $\partial_x \psi(x, t) = v(x, t)$.

Evolution ruled by Burgers equation (inelastic shocks between particles).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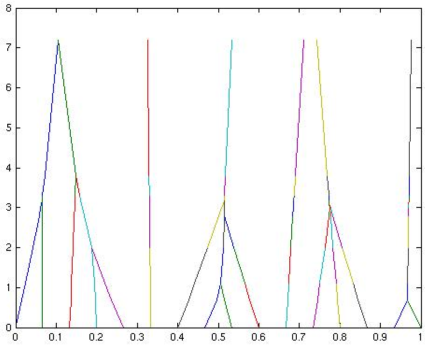
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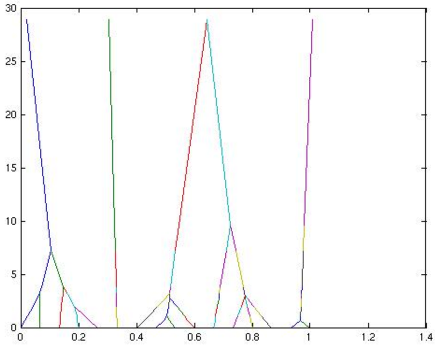
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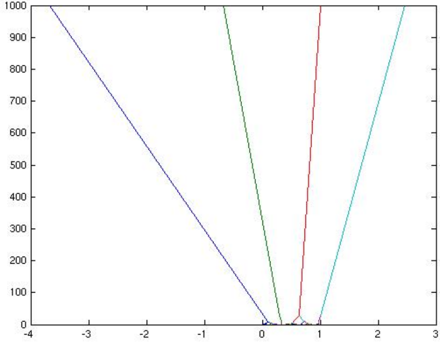
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- Initial data: $X(x) = \psi(x, 0)$, random process.
- $\mathcal{L}_t$: Set of *Lagrangian regular points*, i.e. the points not involved in a shock up to time t.

Hopf (1950), Cole (1951): Explicit solution in terms of $\psi(x, t)$.

Call $f(x) = -x^2/2$. $\mathcal{L}_t$ is the set of points of $\mathrm{ext}^+(X + t^{-1}f)$ that are isolated neither on the left nor on the right.

Theorem

$\mathrm{ext}^+(X + t^{-1}f) \to \mathrm{ext}^+(X)$ *in the Hausdorff metric as* $t \to \infty$.

Thus the set $\mathcal{L}_t$ tends to "shrink" around the points of $\mathrm{ext}^+(X)$.
Explicit description of $\mathcal{L}_t$: Bertoin, Winkel, Giraud,...

Slope of the
concave majorant
Convex hull
of X-t^{-1}f
Initial locations of particles
a
Positions of
particles at time t
X(a)-a^2/2t

Results of negligibility for $\mathrm{ext}^+(X)$

Theorem

Assume we are in one of the following situations:

- X *is a reversible "nice" Markov process, meaning for all* $s > 0$,

  $$\{X(s+t) - X(s) : 0 \leq t < s\} \overset{(d)}{=} \{X(s) - X((s-t)^-) : 0 \leq t < s\}.$$

  (It includes Lévy processes without drift)
- X *is an Ito process which Brownian component vanishes on no open interval.*

Then $\mathrm{ext}(X)$ *has zero* 1*-dimensional Hausdorff measure.*

Topological results

Variation of X on an interval I:

$$V(X,I) = \sup_{x_i,q} \sum_{i=1}^{q} |X(x_i) - X(x_{i+1})|.$$

for $0 < x_1 < \cdots < x_q \in I$.
If X Lévy process,

- either $V(X,I) = \infty$ for all open I,
- or $V(X,I) < \infty$ for all bounded I.

A Lévy process X with bounded variation only varies by jump,

$$X(a) - X(0) - ba = \sum_{0 \leq s \leq a} (X(s^+) - X(s^-)),$$

b: Drift, irrelevant here. We assume $b = 0$.

T_{max}: Time where X approaches its maximal value if I is a compact interval (otherwise, $T_{\max} = +\infty$).

Theorem

- $T_{max} \in \mathrm{ext}^+(X)$,
- $\mathrm{ext}^+(X)$ *is discrete away from* $T_{\max}$,
- $T_{\max}$ *is a left (resp. right) accumulation point of* $\mathrm{ext}^+(X)$ *iff* X *takes strictly positive (resp. negative) values arbitrary close from* 0.

Concave drift

The set $\mathcal{L}_t$ is determined by $\mathrm{ext}^+(X + t^{-1}f)$, where $f(x) = -x^2/2$.

Theorem

X: Levy process without drift and with bounded variation.
f : Concave function.

$$\mathrm{ext}^+(X - f) \subset \mathrm{ext}^+(X) \subset \mathrm{ext}^+(X + f).$$

Call $\overline{g}$ the concave majorant of a function g.
A left (resp. right) accumulation point s in $\mathrm{ext}^+(X + f)$ satisfies

$$\overline{X + f}'(s^-) = f'(s),$$
$$\text{resp.} \overline{X + f}'(s^+) = f'(s).$$

If $f = 0$, you retrieve the fact that every accumulation point s of $\mathrm{ext}^+(X)$ satisfies

$$\overline{X}'(s) = 0,$$

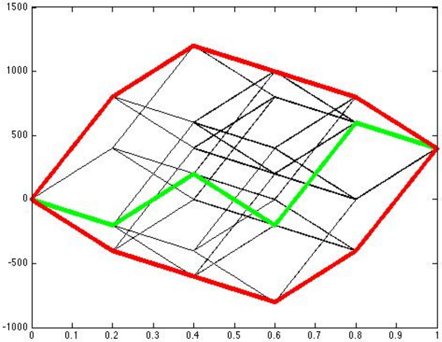
whence $X(s)$ is an extremum of X.

Associated convex body of a 1-dimensional function

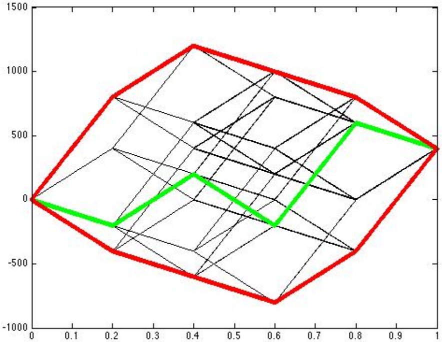
Resource distributed to a population of size N.

- Member labelled k receives r_k.
- Cumulative income function: $f(n) = \sum_{k \leq n} r_k$.

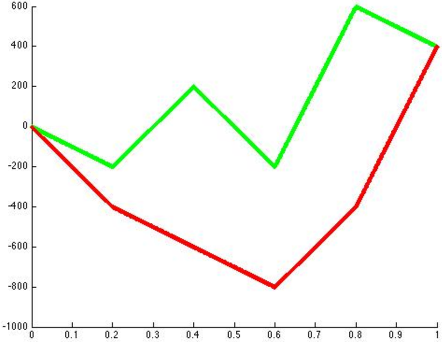
f is extended to a piece-wise linear function on $[0, N]$.



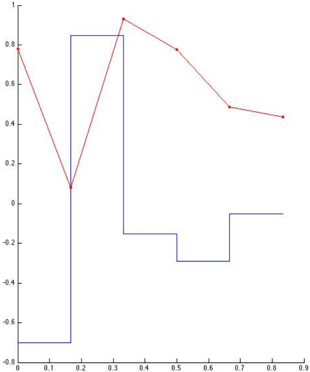
The area of the convex body can measure the inequalities over this particular resource (consider the equality case, where r_k is equal for all k)

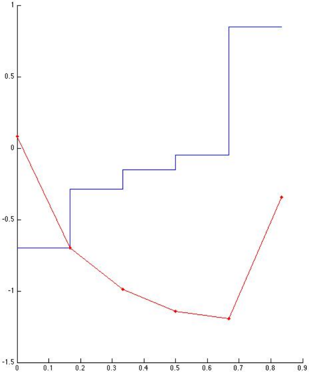


Lower part (red): *convex rearrangement of f*, denoted by $\mathfrak{C}f$.



Rearrangement of the derivative

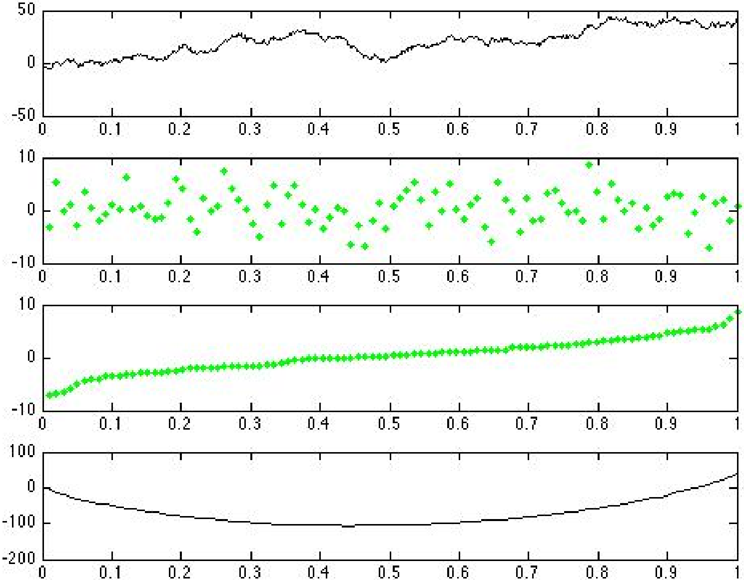




It corresponds to rearranging the derivative in a monotone way. If f' is the derivative of f, and $(\mathfrak{C}f)'$ the derivative of $\mathfrak{C}f$, we have

$$\lambda_1 f'^{-1} = \lambda_1 (\mathfrak{C}f)'^{-1}.$$

Asymptotic rearrangement of the Brownian motion



Theorem (Davydov, Zitikis 2004)

X: Brownian motion.
X_n: Piece-wise linear interpolation of X on $\{0, 1/n, \dots, 1\}$.
Then

$$\sup_{x \in [0,1]} \left| \frac{1}{\sqrt{n}} \mathfrak{C} X_n(x) - L(x) \right| \to 0,$$

L: Lorenz curve.

Other asymptotic convex rearrangements in Davydov & Vershik 1998.

Multidimensional rearrangement

Let $f : [0,1]^d \to \mathbb{R}$, differentiable a.e. such that

$$\int_{[0,1]^d} \|\nabla f(x)\| dx < +\infty.$$

A convex function C is a convex rearrangement of f if

$$\lambda_d \nabla f^{-1} = \lambda_d \nabla C^{-1}.$$

Theorem (Brenier, 91)

Every function f with finite gradient mass has a convex rearrangement. The convex rearrangement is unique up to a constant.

Asymptotic rearrangement

- f: "irregular function"
- f_n: Functions with finite gradient mass, the f_n converge to f. Is there a function C, and positive numbers $\{b_n;\ n \geq 1\}$, such that

$$b_n \mathfrak{C} f_n \to C?$$

If yes, C is a *asymptotic convex rearrangement*.

Theorem

$\{f_n;\ n \geq 1\}$*: Functions with finite gradient mass,*
$\{b_n;\ n \geq 1\}$*: Positive numbers.*
The following assertions are equivalent

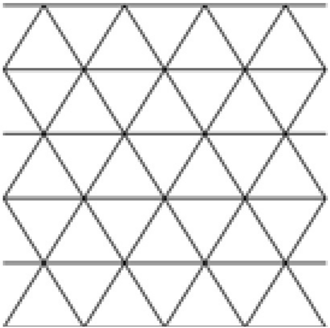
(i) *Weak convergence* $\lambda_d \nabla (b_n f_n)^{-1} \Rightarrow \mu$.
(ii) $b_n \mathfrak{C} f_n(z) \to C(z)$, *for* $z \in \mathrm{int}([0,1]^d)$,
(iii) $\nabla (b_n \mathfrak{C} f_n)^{-1} \to \nabla C$ *in the* L^1 *sense on every sub-compact, whence* $C \in \mathfrak{C} f$.

In this case: $\mu = \lambda_d \nabla C^{-1}$.

Gaussian fields

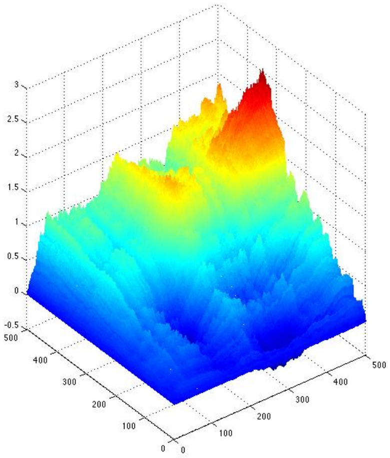
X_n : Approximations of a Gaussian field X on $[0,1]^d$.
X_n is obtained by interpolation of X on a triangulation $\mathcal{T}_n$.
There are regular simplices $T_1, \dots, T_k$, and a discrete group Γ of $\mathbb{R}^d$ such that

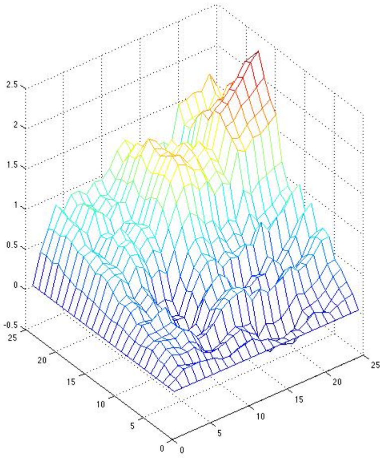
$$\mathcal{T}_n = \{\frac{1}{n}(\gamma + T_j) : \gamma \in \Gamma, 1 \leq j \leq k\}.$$



Call $\mathcal{V}_n$ the set of vertices of $\mathcal{T}_n$.

Brownian sheet approximation





Results

Theorem

Assume the following:
Convergence of the mean: There is a deterministic measure μ *such that, for all Borel set* B*,*

$$\mathbb{E}\int_{[0,1]^d} \mathbf{1}_{\{b_n \nabla X_n(z) \in B\}} dz = \mathbb{E}(\lambda_d (b_n \nabla X_n)^{-1}(B)) \to \mu(B).$$

Θ : *Set of points* (x, y) *where the covariance of* X *is not of class* $\mathcal{C}^1$

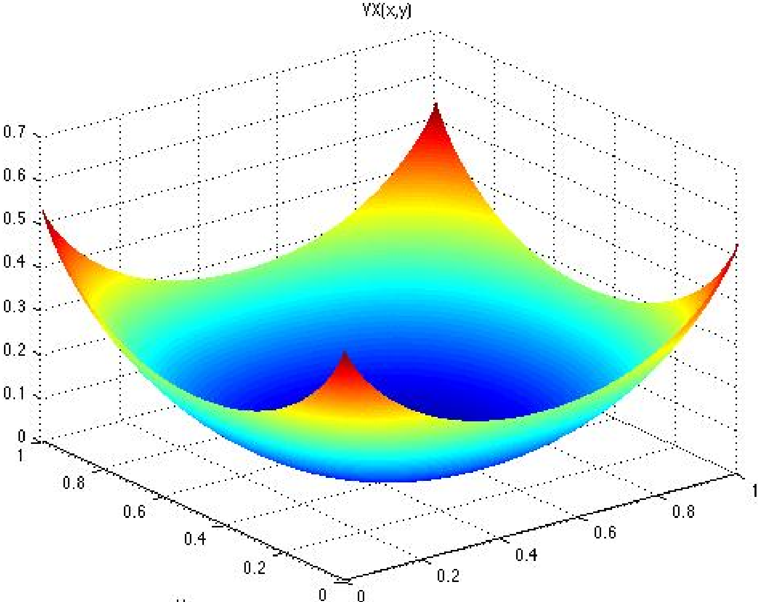
- $\sum_n b_n^{-4} < +\infty$,
- $\sum_n n^{-2d} \mathrm{card}(\Theta \cap \mathcal{V}_n^2) < +\infty$,
- $\sup_{n \in \mathbb{N}, z \in [0,1]^d} \mathrm{cov}(\nabla b_n X_n(z)) < +\infty$.

Then

$$b_n \mathfrak{C} X_n(z) \to C(z), \; z \in \mathrm{int}([0,1]^d)$$

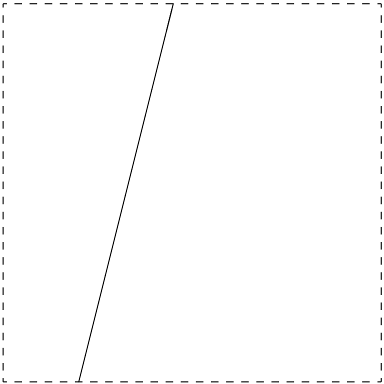
where C *is convex and the gradient distribution of* C *is* μ*.*

Asymptotic rearrangement of the Brownian sheet

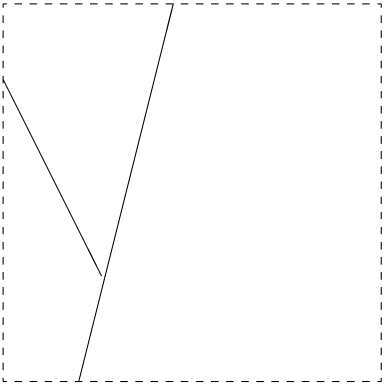


Construction on a bounded convex window

Wait an exponential time with parameter $\mathrm{Per}(W)$. Then draw a random line.



Construction on a bounded convex window W. Each cell created behaves independently. On each cell, wait an exponential time with parameter the perimeter, and then draw a random line inside the cell.



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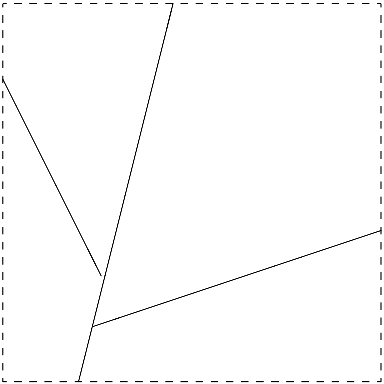
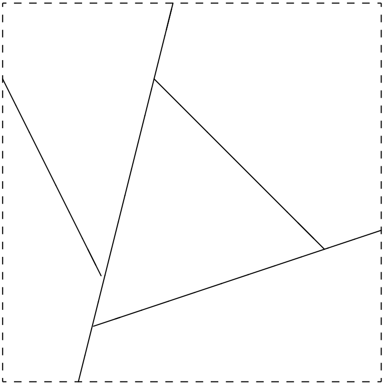
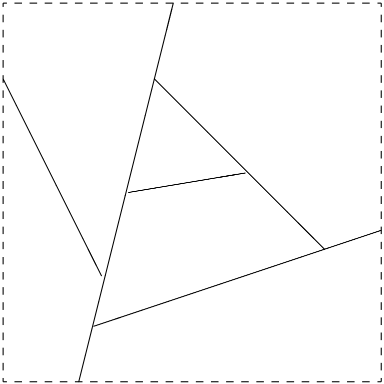
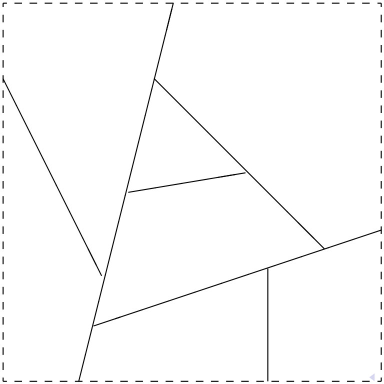






Stop the construction at some predetermined time $a > 0$



Properties of the construction:

- The behaviour of two disjoint cells is independent.
- The process has no memory.

The parameters of the construction are

- The time parameter $a > 0$,
- The law of the random lines.

The same construction can take place in $\mathbb{R}^d$, but lines are replaced by hyperplanes.
It can model the cracking of a drying soil, or the subdivision procedure of cells.

Theorem

There exists a tessellation T on all $\mathbb{R}^d$ such that the law of $T \cap W$, for any compact W, corresponds to the above tessellation.

Such a tessellation is called STIT tessellation.

Mixing property

For two compact sets A and B, and a STIT tessellation T, one has

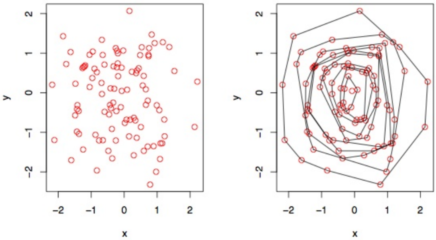
$$\mathbb{P}(A \cap T = \emptyset, (B + h) \cap T = \emptyset) \rightarrow \mathbb{P}(A \cap T = \emptyset)\mathbb{P}(B \cap T = \emptyset),$$

when $\|h\| \rightarrow \infty$.
It implies that T is ergodic. Moreover, one has the decay rate

$$|\mathbb{P}(A \cap T = \emptyset, (B + h) \cap T = \emptyset) - \mathbb{P}(A \cap T = \emptyset)\mathbb{P}(B \cap T = \emptyset)| = o(\|h\|^{-1}).$$

Convex peeling

Convex peeling of a set of points.



For a random set of points X, call $X_{(k)}$ the points that are on the border of the k-th envelope. We have

$$X = \cup_k X_{(k)},$$

and the union is disjoint.
We set

$$\mathrm{env}^k(X) = \mathrm{conv}(X_{(k)}).$$

Weak convergence for α-stable point processes

IID random variables: $\xi_i, i \geq 1$, which common law belongs to the attraction domain of an α-stable law.

$$X_n = \{b_n\xi_1, \ldots, b_n\xi_n\}$$

with proper b_n,

$$X_n \to X,$$

X: α-stable Poisson point process.
Weak convergence:

$$\mathrm{env}^k(X_n) \Rightarrow \mathrm{env}^k(X),$$

The topology is that generated by the Hausdorff metric.
Davydov & Nagaev 2004 deduced the convergence of functionals based on $X_{(k)}$ (extreme summands).

The $\mathcal{F}$-convexity

Closed convex hull

$$\mathrm{conv}(A) = \bigcap_{A \subseteq H} H,$$

for H closed half-space.
Closed half-spaces $\Rightarrow$ arbitrary family of level sets, of the form

$$H = \{x : f(x) \leq \alpha\},$$

for $\alpha \in \mathbb{R}$, f continuous function, $f \in \mathcal{F}$.
Convex hull: $\mathcal{F}$ is the class of linear forms.

Convergence result

Define the $\mathcal{F}$-peeling similarly. We obtained sufficient conditions for having the same result of convergence
There is the weak convergence

$$\mathrm{env}_{\mathcal{F}}^{k}(X_n) \Rightarrow C_{\mathcal{F}}^{k},$$

$C_{\mathcal{F}}^{k} = \mathrm{env}_{\mathcal{F}}^{k}(X)$, for some α-stable Poisson point process X.