Ergodicity of STIT tessellations

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1. Random tessellations

2. STIT tessellations

3. Direct construction of the model
1 Random tessellations

2 STIT tessellations

3 Direct construction of the model
Tessellations

Definition

Tessellation: Set $\mathcal{C} = \{C_1, C_2, \ldots\}$ of convex compact cells, locally finite (for all compact $K$, $\{i; \ C_i \cap K \neq \emptyset\}$ is finite), such that

\[
\mathbb{R}^d = \bigcup_i C_i,
\]

\[
\text{int}(C_i) \cap \text{int}(C_j) = \emptyset, \ i \neq j.
\]

The corresponding closed set is $M = \bigcup_i \partial C_i$. 
Some examples of real structures – Potential applications.

Craquelée on a ceramic (Photo: G. Weil)
cracking simulation (H.-J. Vogel)
Rat muscle tissue (I. Erzen)
Granit joints (D. Nikolayev, S. Siegesmund, S. Mosch, A. Hoffmann)
Gris Perla.jpg
Poisson tessellations

Union of random lines.
\( \Pi \): Point process on \( \mathbb{R}^d \).

\( x \in \Pi, V_x \): Set of points of \( \mathbb{R}^d \) for which \( x \) is the closest element of \( \Pi \),

\[
V_x = \{ y \in \mathbb{R}^2; \| x - y \| = \inf_{x' \in \Pi} \| x' - y \| \}
\]
3D tessellations

3D Poisson hyperplanes tessellation:
Mixed tessellations

(Schmidt, Voss 2010)

- **Grey tessellation**: Low-level servers.
- **Black spots**: High-level servers.
- **Black tessellation** (High-level network): Voronoi tessellation corresponding to black points.

Left: Low-level servers = Voronoi tessellation.
Right: Low-level servers = Poisson line tessellation.
Random tessellations

STIT tessellations

Direct construction of the model
Parameters of the construction

Let $\mathcal{H}$ be the class of hyperplanes of $\mathbb{R}^d$.

- **Intensity**: $a > 0$.
- **Stationary measure** $\nu$ on $\mathcal{H}$, i.e. invariant under the action of translations, and locally finite.
- $W$: Compact window of $\mathbb{R}^d$.

\[
[W] = \{ H \in \mathcal{H} : H \cap W \neq \emptyset \}.
\]

$\nu$ locally finite:

\[
\nu([W]) < +\infty.
\]

Renormalised restriction of $\nu$ to $W$:

\[
\nu_W(\cdot) = \frac{1}{\nu([W])} \nu([W] \cap \cdot).
\]
Start from a bounded window $W$, and after a random exponential time with rate $\nu([W])$, cut the window $W$ by a random line drawn according to $\nu_W$. 
Modelisation of cracking on compact window

Each sub-cell $C$ created behaves independently: it is divided after a random time $\sim \mathcal{E}(\nu([C]))$ by a random line drawn according to $\nu_C$. 

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Each sub-cell $C$ created behaves independantly: it is divided after a random time $\sim \mathcal{E}(\nu([C]))$ by a random line drawn according to $\nu_C$. 
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Each sub-cell $C$ created behaves independently: it is divided after a random time $\sim \mathcal{E} (\nu([C]))$ by a random line drawn according to $\nu_C$. 
Modelisation of cracking on compact window

Stop the process when time $a$ is reached.
Nagel, Weiss, Mecke (Jena)

- Model of cell division.
- Time process without memory (birth and death process).

Call $M_{W,a,\nu}$ the obtained “tessellation”, under the form of the union of the cells boundaries.

$$M_{W,a,\nu} = \bigcup_{C \text{ cell existing at time } a} \partial C.$$  

It is a random element with values in the class $\mathcal{F}(W)$ of closed sets of $W$. $\mathcal{F}(W)$ is endowed with the **Fell topology**, and the corresponding Borel $\sigma$-algebra $\mathcal{B}$. 
Examples (Simulations: J. Ohser)

Simulations of isotropic STIT tessellations. \( \nu \) is stationary and isotropic, i.e. invariant under the action of rotations.

In the isotropic case, the death rate of a cell \( \nu([C]) \) is proportionnal to its perimeter.
Non-isotropic example

Examples where $\nu$ is stationary (but not isotropic).

Here, $\nu([C])$ is proportionnal to the perimeter of the smallest rectangle with sides parallels to the axes containing $C$. 
Binary tree
Tessellation on $\mathbb{R}^d$.

Consistency properties
Nagel and Weiss 2005:
If $W \subseteq W'$, then

$$M_{W,a,\nu} \cap \text{int}(W) \overset{(d)}{=} M_{W',a,\nu} \cap \text{int}(W).$$

Theorem
Let $\{W_i; i \in \mathbb{N}\}$ be a family of compact windows such that

(i) $W_i \uparrow \mathbb{R}^d$,
(ii) $W_i \subset \text{int}(W_{i+1})$.

If a family of random closed sets $\{F_{W_i} \subseteq W_i\}$ satisfy

$$F_{W_i} \cap \text{int}(W_i) \overset{(d)}{=} F_{W_j} \cap \text{int}(W_i), j > i,$$

then there exists a random closed set $F$ of $\mathbb{R}^d$ such that

$$F \cap \text{int}(W_i) \overset{(d)}{=} F_{W_i}, \ i \in \mathbb{N}.$$
There exists a random tessellation $M_{a, \nu} \in \mathcal{F}(\mathbb{R}^d)$ such that

$$(M_{a, \nu} \cap W) \cup \partial W \overset{(d)}{=} M_{a, \nu, W}$$

for all compact $W$. It is the **STIT tessellation with parameters** $a$ and $\nu$. We have

$$a = \mathbb{E} \mathcal{H}^{d-1}(M \cap [0, 1]^d)$$

and $a$ is the intensity.

There exists a direct construction of the tessellation with the help of a point process on $\mathcal{H} \times \mathbb{R}_+$ (marked point process of hyperplanes).
Iteration
Rescaling
Rescaling

\[ \times 2 \Rightarrow \]
Iteration

Let $M, M'$ be two random tessellations.

- $C_1, C_2, \ldots$ cells of $M$.
- $M_1', M_2', \ldots$ independent copies of $M'$, independent of the $C_i$.

Define the iterate of $M$ and $M'$ by

$$M \oplus M' = 2 \bigcup_i \bigcup_{C_j \text{cell of } M'} \partial (C_j \cap C_i).$$

- It is a definition in distribution.
- The operation is not commutative.
Every STIT tessellation $M_{a,\nu}$ satisfies

$$M_{a,\nu} \boxplus M_{a,\nu}^{(d)} = M_{a,\nu}.$$ 

Furthermore, every random tessellation $M$ that satisfies this property is a STIT.
Let $M$ be a stationary tessellation. Define by induction

\[
\begin{align*}
M_1 &= M, \\
M_{n+1} &= M_n \oplus M_n.
\end{align*}
\]

Then

\[M_n \Rightarrow M_{a,\nu},\]

for a certain STIT tessellation $M_{a,\nu}$. 
Mixing properties

A stationary tessellation $M$ is mixing if

$$\mathbb{P}(M \cap K = \emptyset, M \cap (K' + h) = \emptyset) \to \|h\| \to \infty \mathbb{P}(M \cap K = \emptyset) \mathbb{P}(M \cap K' = \emptyset),$$

for all compacts $K, K'$.

**Theorem (L., 2009)**

Let $M$ be a STIT tessellation. Then for all compacts $K$ and $K'$,

$$\mathbb{P}(K \cap M = \emptyset, (K' + h) \cap M = \emptyset) - \mathbb{P}(K \cap M = \emptyset) \mathbb{P}((K' + h) \cap M = \emptyset) = O(1/\|h\|)$$
1 Random tessellations

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3 Direct construction of the model
Poisson point processes, marked with birth times 

\[ \mathbb{R} \times (0, \infty) \]
Poisson point processes, marked with birth times
2. Poisson point processes, marked with birth times

\[ \mathbb{R} \times (0, \infty) \]
Poisson point processes, marked with birth times

\[ \mathbb{R} \times (0, \infty) \]

\( t \)

\( \Psi_t \)

space

time
Now replace $\mathbb{R}$ by $\mathcal{H}$ ... the set of all lines in $\mathbb{R}^2$
Poisson line processes $\Gamma_t$, marked with birth times.
$\Gamma_t$
Poisson point process $\Gamma$ on $\mathcal{H} \times (0, \infty)$ with intensity measure $\nu \times \ell_+$

$\nu$ translation invariant on $\mathcal{H}$

$\ell_+ \ldots$ Lebesgue measure on $(0, \infty)$

For all $t > 0$

$$\Gamma_t = \{ h : (h, s) \in \Gamma : s < t \}$$

is a spatially homogeneous Poisson line process in $\mathbb{R}^2$. 
A preliminary construction
How to start a division of the whole plane when all segments have to have a finite length?
And all nodes are of $T$-type.

The crucial idea (by Joseph Mecke):
Consider the process $(Z_t)_{t>0}$ of the $o$-cells of $(\Gamma_t)_{t>0}$. 
$\Gamma_t$

$Z_t$
For all $t > 0$ the Poisson line process $\Gamma_t$ is a.s. not empty. Assume that the directional distribution $\mathcal{R}$ of the lines is not concentrated in a single point. Then $\Gamma_t$ generates a tessellation with a compact convex polygon $Z_t$ that a.s. contains the origin $o$ in its interior. 

$\implies$ Stochastic process $(Z_t)_{t>0}$ of $o$-cells of $(\Gamma_t)_{t>0}$. 
The isotony
\[ \Gamma_{t_1} \subseteq \Gamma_{t_2} \quad \text{for} \quad t_1 < t_2 \]
implies
\[ Z_{t_1} \supseteq Z_{t_2} \quad \text{for} \quad t_1 < t_2. \]

The process \((Z_t)_{t>0}\) is piecewise constant.
\[ \mathbb{Z} \ldots \text{the set of all integers}. \]

Monotonic sequence \((\sigma_k)_{k \in \mathbb{Z}}\) of times where \((Z_t)_{t>0}\) changes its state.

\(\sigma_k\) is the time when the interior \(\text{int} \, Z_{\sigma_{k-1}}\) is hit by a line from \(\Gamma\).

\[ \ldots < \sigma_{-2} < \sigma_{-1} < \sigma_0 < \sigma_1 < 1 < \sigma_2 < \ldots \]
We obtain

\[ \lim_{k \to -\infty} \sigma_k = 0 \quad \text{and} \quad \lim_{k \to \infty} \sigma_k = \infty \]

Crucial for the construction

\[ Z_{\sigma_k} \uparrow \mathbb{R}^2 \ a.s. \ \text{if} \quad k \downarrow -\infty \]

Also

\[ Z_{\sigma_k} \downarrow \{o\} \ a.s. \ \text{if} \quad k \uparrow \infty \]
A preliminary tessellation of $\mathbb{R}^2$

For $t > 0$ we define a tessellation $\Psi_t$ with the cells

$$Z_t \quad \text{and} \quad \overline{Z_{\sigma_{k-1}} \setminus Z_{\sigma_k}}, \quad \sigma_k < t.$$ 

All these cells are compact, convex and have a pairwise disjoint interior.

Due to

$$Z_{\sigma_k} \uparrow \mathbb{R}^2 \quad \text{a.s. if} \quad k \downarrow -\infty$$

the cells fill the plane, i.e. for all $t > 0$

$$Z_t \cup \bigcup_{\sigma_k < t} \overline{Z_{\sigma_{k-1}} \setminus Z_{\sigma_k}} = \mathbb{R}^2$$
$Z_t$
$Z_t$
\[ Z_t \]

cell of \( \Psi_t \)
cell of $\Psi_t$
cell of $\Psi_t$

$Z_t$
cell of $\Psi_t$

cell of $\Psi_t$  \quad Z_t
cell of $\Psi_t$

cell of $\Psi_t$

$Z_t$
cell of $\Psi_t$

$Z_t$

cell of $\Psi_t$
The random tessellation $\Psi_t$ is non-homogeneous (spatially non-stationary). Intuitively, the older cells of $\Psi_t$,$\overline{Z_{\sigma_k - 1} \setminus Z_{\sigma_k}}$ with $\sigma_k$ close to the time 0 (the moment of the 'Big Bang') are very far from the origin $o \in \mathbb{R}^2$ and they tend to be larger than the younger ones.
The final steps of the construction –
generating a spatially homogeneous random tessellation

For $t > 0$: non-homogeneous tessellation $\Psi_t$ with the cells

$$Z_t \quad \text{and} \quad \overline{Z_{\sigma_k-1} \setminus Z_{\sigma_k}}, \quad \sigma_k < t.$$ 

A cell

$$Z_{\sigma_k-1} \setminus Z_{\sigma_k}$$

is born at the time $\sigma_k < t$.

During the time interval $(\sigma_k, t)$ this bounded cell is divided by random chords as described in the beginning.
Restricted to the cell $\overline{Z_{\sigma_{k-1}} \setminus Z_{\sigma_k}}$ and starting at time $\sigma_k$
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Restricted to the cell \( Z_{\sigma_{k-1}} \setminus Z_{\sigma_k} \) and starting at time \( \sigma_k \).
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Restricted to the cell $Z_{\sigma_{k-1}} \setminus Z_{\sigma_k}$ and starting at time $\sigma_k$

... and finishing at time $t$. 

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Thus the cells $Z_{\sigma_{k-1}} \setminus Z_{\sigma_k}$ are filled during the time interval $(\sigma_k, t)$ such that the resulting tessellation $\Phi_t$

- is spatially homogeneous,
- STIT, i.e. stable under iteration/nesting of tessellations.


Lachièze-Rey R., Mixing properties of STIT tessellations. *Advances in Applied Probability* **43.1** (March 2011)