Rearrangements of Gaussian fields

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1. Introductory example: Brownian motion

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3. Rearrangement of random fields
1 Introductory example: Brownian motion

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Asymptotic rearrangement of the Brownian motion
Theorem (Davydov, Zitikis 2004)

\( X: \) Brownian motion.

\( X_n: \) Piece-wise linear interpolation of \( X \) on \( \{0, 1/n, \ldots, 1\} \).

\( \mathcal{C}X_n: \) Convex rearrangement of \( X_n \).

Then

\[
\sup_{x \in [0,1]} \left| \frac{1}{\sqrt{n}} \mathcal{C}X_n(x) - L(x) \right| \to 0,
\]

\( L: \) Lorenz curve.

Other asymptotic convex rearrangements in Davydov & Vershik 1998.

\( X^H: \) fBm with Hurst parameter \( H \). Then

\[
n^{H-1} \mathcal{C}X_n^H \to L.
\]

(\( L \) is the limit rearrangement for many Gaussian processes with stationary increments)
Convex rearrangement

green: Piecewise linear function $f$.
Lower part (red): convex rearrangement of $f$, denoted by $Cf$. 
Rearrangement of the derivative

It corresponds to rearranging the derivative in a monotone way. If $f'$ is the derivative of $f$, and $(\mathcal{C}f)'$ the derivative of $\mathcal{C}f$, we have

$$\lambda_1 f'^{-1} = \lambda_1 (\mathcal{C}f)'^{-1}.$$
The proof can be decomposed in two steps:

1: The probabilistic result:
Consider the image measure

\[ \mu_n = \lambda_1 \left( n^{-1/2} \nabla X_n \right)^{-1}. \]

Then \( \mu_n \Rightarrow \gamma_1 \) a.s..
(\( \lambda_1 \): 1-dim. Lebesgue, \( \gamma_1 \): Normal distrib., \( \Rightarrow \): weak convergence.)

2: The measure theory result:

**Theorem**

*If a sequence of convex functions \( \{g_n : n \geq 1\} \) satisfies

\[ \lambda_1 (g_n^{-1}) \Rightarrow \mu \]

for some measure \( \mu \) with finite first moment, then \( g_n \rightarrow g \), with \( g \) convex and \( \mu = \lambda_1 g^{-1} \).*
Associated convex body of a 1-dimensional function

Resource distributed to a population of size $N$.

- Member labelled $k$ receives $r_k$.
- Cumulative income function: $f(n) = \sum_{k \leq n} r_k$.

$f$ is extended to a piece-wise linear function on $[0, N]$. 

![Graph of cumulative income function](image.png)
The area of the convex body can measure the inequalities over this particular resource (consider the equality case, where $r_k$ is equal for all $k$)
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Gaussian fields

$X$: Centered Gaussian field, with covariance function

$$
\sigma(z, \zeta) = \mathbb{E}X(z)X(\zeta), \ z, \zeta \in [0, 1]^d.
$$

$X_n$: Approximations of a Gaussian field $X$ on $[0, 1]^d$.

$X_n$ is obtained by interpolation of $X$ on a triangulation $\mathcal{T}_n$.

There are regular simplices $T_1, \ldots, T_k$, and a discrete group $\Gamma$ of $\mathbb{R}^d$ such that

$$
\mathcal{T}_n = \left\{ \frac{1}{n}(\gamma + T_j) : \gamma \in \Gamma, 1 \leq j \leq k \right\}.
$$
Brownian sheet approximation
Results

Define

\[ \mu_n = \lambda_d (b_n \nabla X_n)^{-1} \]

and

\[ \sigma_{z,\zeta}^{(2)}(u, v) = \sigma(z, \zeta) + \sigma(z + u, \zeta + v) - \sigma(z + u, \zeta) - \sigma(z, \zeta + v), \]

the second order local increment of \( \sigma \).

Theorem

Assume the following: For all \( u, v \) in \( \mathbb{R}^d \)

\[ (nb_n)^2 \sigma_{z,z}^{(2)}(n^{-1} u, n^{-1} v) \rightarrow \sigma_z^{\text{diag}}(u, v) \]

uniformly in \( z \in [0, 1]^d \).

Then there is a deterministic measure \( \mu \) such that, for all Borel set \( B \),

\[ \mathbb{E} \int_{[0,1]^d} \mathbf{1}_{\{b_n \nabla X_n(z) \in B\}} dz = \mathbb{E}(\mu_n(B)) \rightarrow \mu(B). \]
examples

Multifractional Brownian field:

\[
\sigma(z, \zeta) = \|z\|^\alpha + \|\zeta\|^\alpha - \|z - \zeta\|^\alpha, \quad \alpha \in (0, 2)
\]

\[
\begin{cases}
\sigma^{(2)}_{z, z}(u, v) = \|u\|^\alpha + \|v\|^\alpha - \|u - v\|^\alpha = \sigma_{z, z}^{\text{diag}}(u, v), \\
b_n = n^{\alpha/2 - 1}
\end{cases}
\]

Brownian sheet:

\[
\sigma(z, \zeta) = \prod_i \min(z_i, \zeta_i).
\]

\[
\begin{cases}
\sigma_{z, z}^{\text{diag}}(u, v) = \langle l(z), u \wedge v - u \wedge 0 - v \wedge 0 \rangle, \\
b_n = \sqrt{n}
\end{cases}
\]

with

\[
l(z) = (z_2 \ldots z_d, z_1 z_3 \ldots z_d, \ldots, z_1 \ldots z_{d-1}).
\]
\( \varphi_n \): Characteristic function of \( \mu_n \).

**Theorem**

Let \( h \in \mathbb{R}^d \).

\[
\mathbb{E} \left| \varphi_n(h) - \mathbb{E} \varphi_n(h) \right|^4 \leq C \left( \frac{n}{b_n} \right)^2 \sum_{S,S' \in T_n} \text{vol}(S) \text{vol}(S') \left| \sigma_{z,\zeta}^{(2)}(n^{-1}u, n^{-1}v) \right| \right)^2
\]

\((u, v \text{ are the directions of edges of resp. } S \text{ and } S').\)
For the Multivariate Brownian field and the Brownian sheet, the right hand member is in $O(n^{-2})$, whence (Borel-Cantelli),

$$\mu_n \Rightarrow \mu$$

a.s..

Remarks:

- $\mu$ is deterministic,
- the convergence happens on each sample path.

New consistent estimators for parameters $\sigma(z, \zeta)$:

- Regularity parameters (Hurst Index),
- Directional parameters (Privileged axes)
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Multidimensional rearrangement

Let $f : [0, 1]^d \to \mathbb{R}$, differentiable a.e. such that

$$\int_{[0,1]^d} \| \nabla f(x) \| \, dx < +\infty.$$

A convex function $C$ is a convex rearrangement of $f$ if

$$\lambda_d \nabla f^{-1} = \lambda_d \nabla C^{-1}.$$

**Theorem (Brenier, 91)**

*Every function $f$ with finite gradient mass has a convex rearrangement $Cf$. The convex rearrangement is unique up to a constant.*
Asymptotic rearrangement

- \( f \): “irregular function”
- \( f_n \): Functions with finite gradient mass, the \( f_n \) converge to \( f \). Is there a function \( C \), and positive numbers \( \{ b_n; n \geq 1 \} \), such that
  \[
  b_n \mathcal{C} f_n \to C?
  \]

If yes, \( C \) is an asymptotic convex rearrangement.

Theorem

\( \{ f_n; n \geq 1 \} \): Functions with finite gradient mass,
\( \{ b_n; n \geq 1 \} \): Positive numbers.

The following assertions are equivalent

(i) Weak convergence \( \lambda_d \nabla (b_n f_n)^{-1} \Rightarrow \mu \).
(ii) \( b_n \mathcal{C} f_n(z) \to C(z) \), for \( z \in \text{int}(\mathbb{I}^d) \),
(iii) \( \nabla (b_n \mathcal{C} f_n)^{-1} \to \nabla C \) in the \( L^1 \) sense on every sub-compact, whence \( C \in \mathcal{C} f \).

In this case: \( \mu = \lambda_d \nabla C^{-1} \).
Asymptotic rearrangement of the Brownian sheet

\[ n^{-1/2} \mathcal{C} X_n(z) \to C(z) \quad \text{a.s.,} \quad z \in (0, 1)^2, \]