Weak invariance principle and exponential bounds for some special functions of intermittent maps

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Abstract

We consider a parametric class T_γ of expanding maps of $[0,1]$ with a neutral fixed point at 0 for which there exists an unique invariant absolutely continuous probability measure ν_γ on $[0,1]$. On the probability space $([0,1], \nu_\gamma)$, we prove the weak invariance principle for the partial sums of $f \circ T_\gamma^i$ in some special cases involving non-standard normalization. We also prove new moment inequalities and exponential bounds for the partial sums of $f \circ T_\gamma^i$ when f is some Hölder function such that $f(0) = \nu_\gamma(f)$.

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1 Introduction

For γ in $]0,1[$, we consider the following intermittent map T_γ from $[0,1]$ to $[0,1]$, introduced in Liverani, Saussol and Vaienti (1999):

$$T_\gamma(x) = \begin{cases} x(1+2^\gamma x^\gamma) & \text{if } x \in [0,1/2] \\ 2x-1 & \text{if } x \in (1/2,1] \end{cases}$$

We denote by ν_γ the unique T_γ-invariant probability measure on $[0,1]$ which is absolutely continuous with respect to the Lebesgue measure.

In 1999, Young showed that such systems (among many others) may be described by a Young tower with polynomial decay of the return time. From this construction, she was able

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to control the covariances $\nu_\gamma(g \circ T^n \cdot (f - \nu_\gamma(f)))$ for any bounded function g and any Hölder function f, and then to prove that, on the probability space $([0,1], \nu_\gamma)$,

$$\frac{S_n(f)}{\sqrt{n}} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (f \circ T_\gamma^i - \nu_\gamma(f))$$

converges in distribution to a normal law as soon as $\gamma < 1/2$. Note that, in that case, one can easily prove that the weak invariance principle holds, which means that the normalized partial sum process converges in distribution to a Wiener process in the Skorohod topology.

In his (2004a) paper, Gouëzel has given a complete picture of the limit behaviour of the distribution of $S_n(f)$ when f is any Hölder function. If $\gamma = 1/2$ and $f(0) \neq \nu_{1/2}(f)$, he proved that the central limit theorem remains true with the normalization $\sqrt{n \ln(n)}$. When $1/2 < \gamma < 1$ and $f(0) \neq \nu_\gamma(f)$, he proved that $n^{-\gamma} S_n(f)$ converges in distribution to a stable law. If $f(0) = \nu_\gamma(f)$ and $|f(x) - f(0)| \leq C x^a$, he proved that the central limit theorem holds with the normalization $\sqrt{n}$ provided that $\gamma < a + 1/2$. Gouëzel studied also the case where $f(x) = x^{(2\gamma-1)/2}$ for $\gamma < 1/2$, and proved that the central limit theorem holds for the normalization $\sqrt{n \ln(n)}$.

In this note, we shall prove that in every situation described by Gouëzel for which the central limit theorem holds, the weak invariance principle also holds (with the appropriate normalization). Moreover, we shall give some new moment inequalities and exponential bounds for $S_n(f)$ in the special case where $f(0) = \nu_\gamma(f)$.

To prove our results, we shall first introduce an appropriate Markov chain as follows. Let K_γ be the Perron-Frobenius operator of T_γ with respect to ν_γ: for any bounded measurable functions f, g,

$$\nu_\gamma(f \cdot g \circ T_\gamma) = \nu_\gamma(K_\gamma(f) g) \,. \tag{1.1}$$

Let $(Y_i)_{i \geq 0}$ be a stationary Markov chain with invariant measure ν_γ and transition Kernel K_γ. It is well known (see for instance Lemma XI.3 in Hennion and Hervé (2001)) that on the probability space $([0,1], \nu_\gamma)$, the random variable $(T_\gamma, T_\gamma^2, \ldots, T_\gamma^n)$ is distributed as $(Y_n, Y_{n-1}, \ldots, Y_1)$.

To prove the weak invariance principle, we shall apply the sharp results given in Merlevède and Peligrad (2006) to the normalized partial sum process of the sequence $(f(Y_i) - \nu_\gamma(f))_{i \geq 0}$. To prove the moment (resp. exponential) inequalities, the main point is to control the quantity $\|K_\gamma^n(f) - \nu_\gamma(f)\|_{p,\nu_\gamma}$ (resp. $\|K_\gamma^n(f) - \nu_\gamma(f)\|_{\infty,\nu_\gamma}$) when $f(0) = \nu_\gamma(f)$, and next to apply the Burkholder inequality (resp. Hoeffding inequality) given in Peligrad *et al.* (2007) to the sums $\sum_{i=1}^n (f(Y_i) - \nu_\gamma(f))$.

2 Weak invariance principle when $\gamma = 1/2$.

Let $\gamma = 1/2$. According to Item 2 of the comments following Theorem 1.3 in Gouëzel (2004a), we know that, for any Hölder function f,

$$\frac{1}{\sqrt{n \ln(n)}} S_n(f) \text{ converges in distribution to } \sqrt{h(1/2)}(f(0) - \nu_{1/2}(f))N, \tag{2.1}$$

where N is a standard Gaussian. Moreover, if $f(0) = \nu_{1/2}(f)$, $n^{-1/2} S_n(f)$ converges in distribution to a normal law.

In the next theorem, we show that the weak invariance principle also holds. Moreover, we show that if $f(0) = \nu_{1/2}(f)$, the limiting variance is the usual covariance series.

Theorem 2.1. *Let $\gamma = 1/2$ and let f be any Hölder function. Let W be a standard Brownian motion.*

1. *On the probability space $([0,1], \nu_{1/2})$, the process*

$$\Big\{ \frac{1}{\sqrt{n \ln(n)}} S_{[nt]}(f), t \in [0,1] \Big\}$$

converges in distribution to $\sqrt{h(1/2)}(f(0) - \nu_{1/2}(f))W$, in the Skorohod topology.

2. *If $f(0) = \nu_{1/2}(f)$, then the series*

$$\sigma^2(f) = \nu_{1/2}((f - \nu_{1/2}(f))^2) + 2 \sum_{k>0} \nu_{1/2}((f - \nu_{1/2}(f)) f \circ T^k)$$

converge absolutely to some nonnegative number. In addition, on the probability space $([0,1], \nu_{1/2})$, the process

$$\Big\{ \frac{1}{\sqrt{n}} S_{[nt]}(f), t \in [0,1] \Big\}$$

converges in distribution to $\sigma(f)W$, in the Skorohod topology.

3 Weak invariance principle for $f(x) = x^{(2\gamma-1)/2}$.

Let $\gamma < 1/2$, and let f be the function from $]0,1]$ to $\mathbb{R}^+$ defined by $f(x) = x^{(2\gamma-1)/2}$. From the comment 3 page 88-89 in Gouëzel (2004a), we know that

$$\frac{1}{\sqrt{n \ln(n)}} S_n(f) \text{ converges in distribution to } \sqrt{h(1/2)} 2^{(1-2\gamma)/2} N, \tag{3.1}$$

where N is a standard Gaussian (the limiting variance was communicated to us by S. Gouëzel and can be obtained by following the arguments given in the proof of his Theorem 1.3).

In the next theorem, we show that the weak invariance principle also holds.

Theorem 3.1. *Let $\gamma < 1/2$ and $f(x) = x^{(2\gamma-1)/2}$. Let W be a standard Brownian motion. On the probability space $([0,1], \nu_\gamma)$, the process*

$$\Big\{\frac{1}{\sqrt{n\ln(n)}}S_{[nt]}(f), t \in [0,1]\Big\}$$

converges in distribution to $\sqrt{h(1/2)}2^{(1-2\gamma)/2}W$, in the Skorohod topology.

4 On the functions such that $f(0) = \nu_\gamma(f)$.

As in Gouëzel (2004a), our results will depend on the behaviour of f around 0. Therefore, we first introduce the following class:

Definition 4.1. For any $\gamma \in]0,1[$ and any $a > 0$, let $\mathcal{H}_{0,\gamma,a}$ be the class of Hölder functions f on $[0,1]$ such that $f(0) = \nu_\gamma(f)$ and $|f(x) - f(0)| \leq Cx^a$.

In his Theorem 2.4.14, Gouëzel (2004c) proved that: for any $\gamma \in]0,1[$ any $a > 0$ and any f in $\mathcal{H}_{0,\gamma,a}$, there exists a positive constant C_1 such that

$$\|K_\gamma^n(f) - \nu_\gamma(f)\|_{1,\nu_\gamma} \leq C_1 \max\Big(\frac{1}{n^{1/\gamma}}, \frac{1}{n^{(1+a-\gamma)/\gamma}}\Big). \tag{4.1}$$

In the next proposition, we shall give an upper bound for the $\mathbb{L}^\infty(\nu_\gamma)$-norm.

Proposition 4.2. *For any $\gamma \in]0,1[$ any $a > 0$ and any f in $\mathcal{H}_{0,\gamma,a}$, there exists a positive constant C_∞ such that*

$$\|K_\gamma^n(f) - \nu_\gamma(f)\|_{\infty,\nu_\gamma} \leq C_\infty \max\Big(\frac{1}{n^{a/\gamma}}, \frac{1}{n}\Big).$$

Remark 4.3. *Combining (4.1) and Proposition 4.2, we obtain that, for any $p \in [1,\infty]$, there exists a positive constant C_p such that*

$$\|K_\gamma^n(f) - \nu_\gamma(f)\|_{p,\nu_\gamma} \leq \frac{C_p}{n^{(1-\gamma)/(p\gamma)}} \max\Big(\frac{1}{n^{a/\gamma}}, \frac{1}{n}\Big).$$

Starting from Remark 4.3 and applying the moment inequality given in Peligrad *et al.* (2007), we obtain the following results:

Theorem 4.4. *For any $a > 0$, any f in $\mathcal{H}_{0,\gamma,a}$ and any $p \in [2,\infty[$, we have*

1. *If $0 < \gamma < 2(ap+1)/(p+2)$, then there exists some positive constant C such that*

$$\Big\|\max_{1\leq k\leq n}|S_k(f)|\Big\|_{p,\nu_\gamma} \leq C\sqrt{n}\,.$$

2. *If $\gamma = 2(ap+1)/(p+2)$, then there exists some positive constant C such that*

$$\Big\| \max_{1\le k\le n} |S_k(f)| \Big\|_{p,\nu_\gamma} \le C\sqrt{n}\ln(n)\,.$$

3. *If $2(ap+1)/(p+2) < \gamma < 1$, then there exists some positive constant C such that*

$$\Big\| \max_{1\le k\le n} |S_k(f)| \Big\|_{p,\nu_\gamma} \le Cn^{(\gamma(p+1)-ap-1)/p\gamma}\,.$$

Of course, this result is no longer true if $p = \infty$. Instead, we have the following exponential bounds:

Theorem 4.5. *For any $a > 0$ and any f in $\mathcal{H}_{0,\gamma,a}$, we have*

1. *If $0 < \gamma < 2a$, then there exists two positive constants C_1 and C_2 such that, for any $x > 0$,*
$$\nu_\gamma\left(\max_{1\le k\le n} |S_k(f)| \ge x\sqrt{n}\right) \le C_1\exp(-C_2x^2)\,.$$

2. *If $\gamma = 2a$, then there exists two positive constants C_1 and C_2 such that, for any $x > 0$,*

$$\nu_\gamma\left(\max_{1\le k\le n} |S_k(f)| \ge x\sqrt{n}\ln(n)\right) \le C_1\exp(-C_2x^2)\,.$$

3. *If $2a < \gamma < 1$, then there exists two positive constants C_1 and C_2 such that, for any $x > 0$,*
$$\nu_\gamma\left(\max_{1\le k\le n} |S_k(f)| \ge xn^{(\gamma-a)/\gamma}\right) \le C_1\exp(-C_2x^2)\,.$$

Remark 4.6. *As a straightforward consequence of Theorem 4.5, we obtain that*

1. *If $0 < \gamma < 2a$, then there exists a positive constant C such that*

$$\limsup_{n\to\infty} \frac{|S_n(f)|}{\sqrt{n\ln(\ln(n))}} \le C \quad \textit{almost everywhere.}$$

2. *If $\gamma = 2a$, then there exists a positive constant C such that*

$$\limsup_{n\to\infty} \frac{|S_n(f)|}{\ln(n)\sqrt{n\ln(\ln(n))}} \le C \quad \textit{almost everywhere.}$$

3. *If $2a < \gamma < 1$, then there exists a positive constant C such that*

$$\limsup_{n\to\infty} \frac{|S_n(f)|}{n^{(\gamma-a)/\gamma}\sqrt{\ln(\ln(n))}} \le C \quad \textit{almost everywhere.}$$

As recalled in the introduction, Gouëzel (2004a) has proved that if f belongs to $\mathcal{H}_{0,\gamma,a}$ for $0 < \gamma < a + 1/2$ then $n^{-1/2} S_n(f)$ converges to a normal distribution. In the next theorem, we show that the weak invariance principle also holds, and that the limiting variance is the usual covariance series. Note that this result is more precise than Item 1 of Theorem 4.4 in the case where $p = 2$.

Theorem 4.7. *Let W be a standard Brownian motion. For any $a > 0$, any $0 < \gamma < a + 1/2$ and any f in $\mathcal{H}_{0,\gamma,a}$, the series*

$$\sigma^2(f) = \nu_\gamma((f - \nu_\gamma(f))^2) + 2 \sum_{k>0} \nu_\gamma((f - \nu_\gamma(f)) f \circ T^k) \tag{4.2}$$

converges absolutely. Moreover, on the probability space $([0,1], \nu_\gamma)$, the process

$$\Big\{ \frac{1}{\sqrt{n}} S_{[nt]}(f), t \in [0,1] \Big\}$$

converges in distribution to $\sigma(f)W$, in the Skorohod topology.

5 Proofs

From now, C and D are positive constants which may vary from line to line.

5.1 Proof of Theorem 2.1

We first note that Item 2 of Theorem 2.1 is a consequence of Theorem 4.7 (if $\gamma = 1/2$, the constraint $\gamma < a + 1/2$ is clearly satisfied), which will be proved in Section 5.5. Now, if $f(0) = \nu_{1/2}(f)$, then Item 1 is a straightforward consequence of Item 2. Consequently, it remains to prove Item 1 in the case where $f(0) \neq \nu_{1/2}(f)$.

Let $X_i = f(Y_i) - \nu_{1/2}(f)$, where $(Y_i)_{i \in \mathbb{Z}}$ is the Markov chain with transition Kernel $K_{1/2}$ and invariant measure $\nu_{1/2}$. Recall that $(T_{1/2}, T_{1/2}^2, \ldots, T_{1/2}^n)$ is distributed as $(Y_n, Y_{n-1}, \ldots, Y_1)$. Let $S_n = \sum_{i=1}^n X_i$, and let $c(f) = \sqrt{h(1/2)}(f(0) - \nu_{1/2}(f))$. To prove Item 1, we shall prove that

$$\Big\{ \frac{1}{\sqrt{n \ln(n)}} S_{[nt]}, t \in [0,1] \Big\} \tag{5.1}$$

converges in distribution to $c(f)W$, in the Skorohod topology. To see that this result implies Item 1 of Theorem 2.1, it suffices to notice that the process $W_n(f) = \{W_n(f,t), t \in [0,1]\}$ defined by

$$W_n(f,t) = \frac{1}{\sqrt{n \ln(n)}} \sum_{k=1}^{[nt]} (f \circ T^{n-k+1} - \nu_{1/2}(f)) + \frac{nt - [nt]}{\sqrt{n \ln(n)}} (f \circ T^{n-[nt]} - \nu_{1/2}(f)),$$

converges in distribution in $C([0,1], \| \cdot \|_\infty)$ to $c(f)W$, so that $W_n(f,1) - W_n(f)$ converges in distribution in $C([0,1], \|\cdot\|_\infty)$ to $c(f)(W(1)-W)$. Hence $\{W_n(f,1)-W_n(f,1-t), t\in[0,1]\}$ converges in distribution in $C([0,1], \|\cdot\|_\infty)$ to $\{c(f)(W(1)-W(1-t)), t\in[0,1]\}$ which is distributed as $c(f)W$. Now $\{W_n(f,1)-W_n(f,1-t), t\in[0,1]\}$ is equal to the process

$$\Big\{\frac{1}{\sqrt{n\ln(n)}}S_{[nt]}(f)+\frac{nt-[nt]}{\sqrt{n\ln(n)}}(f\circ T^{[nt]+1}-\nu_{1/2}(f)), t\in[0,1]\Big\},$$

which consequently converges in distribution in $C([0,1], \|\cdot\|_\infty)$ to $c(f)W$. Theorem 2.1 easily follows.

To prove the weak convergence of the process (5.1), we use Corollary 3 in Merlevède and Peligrad (2006). Let $B_n=\sqrt{\pi/2}\mathbb{E}(|S_n|)$. Applying this corollary to the bounded random variables X_i, we infer that if

$$\sigma_n^2=\mathbb{E}(S_n^2)\to\infty\,, \tag{5.2}$$

$$\sum_{i=1}^n i\|\mathbb{E}(X_i|Y_0)\|_1=o(\sigma_n^2)\,, \tag{5.3}$$

and

$$\lim_{n\to\infty}\sigma_n^{-2}\mathbb{E}(S_n^2|Y_{-n})=1 \text{ in } \mathbb{L}^1\,, \tag{5.4}$$

are satisfied, then the process $\{B_n^{-1}S_{[nt]}, t\in[0,1]\}$ converges in distribution to W, in the Skorohod topology. We shall see in the rest of the proof that necessarily,

$$B_n\sim\sqrt{h(1/2)}|f(0)-\nu_{1/2}(f)|\sqrt{n\ln(n)}\,. \tag{5.5}$$

It remains to prove (5.2), (5.3) and (5.4). We first recall that from Young (1999), if f is δ-Hölder for some δ in $]0,1]$,

$$|\nu_{1/2}(g\cdot K_{1/2}^n(f-\nu_{1/2}(f)))|\le\frac{C}{n}\|g\|_\infty\mathcal{L}_\delta(f)\,, \tag{5.6}$$

where

$$\mathcal{L}_\delta(f)=\sup_{x,y\in[0,1]}\frac{|f(x)-f(y)|}{|x-y|^\delta}\,.$$

Clearly, Inequality (5.6) is equivalent to

$$\|\mathbb{E}(X_n|Y_0)\|_1=\|K_{1/2}^n(f)-\nu_{1/2}(f)\|_1\le\frac{C}{n}\mathcal{L}_\delta(f)\,. \tag{5.7}$$

Since $\sigma_n^2\le n\|f\|_\infty(\|X_0\|_1+2\sum_{i=1}^n\|\mathbb{E}(X_n|Y_0)\|_1)$, we obtain from (5.7) that

$$\sigma_n^2\le C\|f\|_\infty n\ln(n)\,. \tag{5.8}$$

Clearly, (5.8) implies that $\{S_n/\sqrt{n\ln(n)}\}$ is uniformly integrable. Consequently, using (2.1) and the fact that $\{|S_n|/\sqrt{n\ln(n)}\}$ is uniformly integrable, we derive that (5.5) holds. Since

$f(0) \neq \nu_{1/2}(f)$, it follows that for n large enough,

$$\sigma_n^2 \geq (\mathbb{E}(|S_n|))^2 \geq Cn \ln(n) \,, \tag{5.9}$$

for some $C > 0$, so that (5.2) is satisfied.

Now, combining (5.7) and (5.9), we infer that (5.3) holds. It remains to prove (5.4). According to Inequality (4.92) in Merlevède and Peligrad (2006), we get that

$$\sigma_n^{-2} \|\mathbb{E}(S_n^2|Y_{-n}) - \mathbb{E}(S_n^2)\|_1 \leq 2\sigma_n^{-2} \sum_{i=n+1}^{2n} \sum_{j=i}^{2n} \|\mathbb{E}(X_i X_j|Y_0) - \mathbb{E}(X_i X_j)\|_1 \,. \tag{5.10}$$

Let $f^{(0)} = f - \nu_{1/2}(f)$. For $j \geq i$,

$$\|\mathbb{E}(X_i X_j|Y_0) - \mathbb{E}(X_i X_j)\|_1 = \nu_{1/2}\Big(\big|K^i(f^{(0)} K^{j-i} f^{(0)}) - \nu_{1/2}\big(K^i(f^{(0)} K^{j-i} f^{(0)})\big)\big|\Big) \,.$$

According to Lemmas 2.1 and 2.2 in Dedecker and Prieur (2008), we have that

$$\nu_{1/2}\Big(\big|K^i(f^{(0)} K^{j-i} f^{(0)}) - \nu_{1/2}\big(K^i(f^{(0)} K^{j-i} f^{(0)})\big)\big|\Big) \leq \frac{C}{n} \mathcal{L}_\delta(f) \,.$$

These considerations together with (5.9) end the proof of (5.4).

5.2 Proof of Proposition 3.1

We use the same notations as in the proof of Theorem 2.1: $(Y_i)_{i\in\mathbb{Z}}$ is the Markov chain with transition operator K_γ and invariant measure ν_γ, and $X_i = f(Y_i) - \nu_\gamma(f)$. We use again Corollary 3 in Merlevède and Peligrad (2006). We still have to prove (5.2) and (5.4). Since the variables are not bounded, instead of (5.3) we have to prove that

$$\sum_{i=1}^{n} i \int_0^{\|\mathbb{E}(X_i|Y_0)\|_1} Q_f \circ G_f(u) du = o(\sigma_n^2) \,, \tag{5.11}$$

where $Q_f(u) = \inf\{t \geq 0, \nu_\gamma(f > t) \leq u\}$ and G_f is the inverse function of $x \mapsto \int_0^x Q_f(u) du$. Note that $Q_f(u) = (F_\gamma^{-1}(u))^{(2\gamma-1)/2}$ where $F_\gamma(t) = \nu_\gamma([0,t])$. Since the density h of ν_γ is such that $ax^{-\gamma} \leq h(x) \leq bx^{-\gamma}$ (see Section 5.3), we derive that $C_1 u^{\frac{2\gamma-1}{2(1-\gamma)}} \leq Q_f(u) \leq C_2 u^{\frac{2\gamma-1}{2(1-\gamma)}}$, so that $G_f(u) \geq Cu^{2(1-\gamma)}$. Hence $Q_f \circ G_f(u) \leq Cu^{2\gamma-1}$ and to prove (5.11), it remains to show that

$$\sum_{i=1}^{n} i \|\mathbb{E}(X_i|Y_0)\|_1^{2\gamma} = o(\sigma_n^2) \,. \tag{5.12}$$

Here, we need the following definition:

Definition 5.1. For any integrable real-valued random variable X, let $X^{(0)} = X - \mathbb{E}(X)$. For any random variable $Y = (Y_1, \cdots, Y_k)$ with values in $\mathbb{R}^k$ and any σ-algebra $\mathcal{F}$, let

$$\alpha(\mathcal{F}, Y) = \sup_{(x_1,\ldots,x_k)\in\mathbb{R}^k} \left\| \left(\mathbb{E}\Big(\prod_{j=1}^{k} (\mathbb{1}_{Y_j \le x_j})^{(0)} \Big| \mathcal{F} \Big) \right)^{(0)} \right\|_1 .$$

For the Markov chain $\mathbf{Y} = (Y_i)_{i\in\mathbb{Z}}$, we then define

$$\alpha_{k,\mathbf{Y}}(n) = \max_{1\le l\le k} \sup_{i_l > \ldots > i_1 \ge n} \alpha(\sigma(Y_0), (Y_{i_1}, \ldots, Y_{i_l})). \tag{5.13}$$

In Proposition 1.12 of Dedecker *et al.* (2008), it is proved that $\alpha_{k,\mathbf{Y}}(n) \le C(k,\gamma) n^{(\gamma-1)/\gamma}$. Since f is monotonic, the coefficients of the sequence $(f(Y_i))_{i\in\mathbb{Z}}$ are smaller than that of $(Y_i)_{i\in\mathbb{Z}}$. Hence, applying Theorem 1.1 in Rio (2000), one has

$$\|\mathbb{E}(X_n|Y_0)\|_1 \le 2 \int_0^{\alpha_{1,\mathbf{Y}}(n)} Q_f(u) du \le \frac{C}{n^{1/(2\gamma)}} .$$

Hence to prove (5.12), it suffices to show that (5.9) holds. We proceed as in the proof of Theorem 2.1. First, applying again Theorem 1.1 in Rio (2000), one has

$$|\mathrm{Cov}(X_0, X_n)| \le 2 \int_0^{\alpha_{1,\mathbf{Y}}(n)} Q_f^2(u) du \le C n^{-1} ,$$

so that $\sigma_n^2 \le C n \ln(n)$. Consequently $\{|S_n|/\sqrt{n\ln(n)}\}$ is uniformly integrable. Using (3.1), we derive that $B_n \sim \sqrt{h(1/2)} 2^{(1-2\gamma)/2} \sqrt{n \ln(n)}$. Hence (5.9) holds, so that (5.2) and (5.11) are satisfied.

To complete the proof, it remains to prove (5.4). Let us first prove that, for $j > i > 0$,

$$\|\mathbb{E}(X_i X_j | Y_0) - \mathbb{E}(X_i X_j)\|_1 \le 16 \int_0^{\alpha_{2,\mathbf{Y}}(i)/4} Q_f^2(u) du. \tag{5.14}$$

Setting $A := \mathrm{sign}\{\mathbb{E}(X_i X_j|Y_0) - \mathbb{E}(X_i X_j)\}$, we have that

$$\|\mathbb{E}(X_i X_j|Y_0) - \mathbb{E}(X_i X_j)\|_1 = \mathbb{E}\Big\{ A\Big(\mathbb{E}(X_i X_j|Y_0) - \mathbb{E}(X_i X_j) \Big) \Big\} = \mathbb{E}\big((A - \mathbb{E}(A)) X_i X_j \big) .$$

From Proposition A.1 and Lemma A.1 in Dedecker and Rio (2008), noticing that $Q_A(u) \le 1$, we have that

$$\mathbb{E}((A - \mathbb{E}(A)) X_i X_j) \le 16 \int_0^{\bar\alpha(A, X_i, X_j)/2} Q_f^2(u) du ,$$

where for real valued random variables A, U, V,

$$\bar\alpha(A, U, V) = \sup_{(s,t,u)\in\mathbb{R}^3} \big| \mathbb{E}((\mathbb{1}_{A\le s} - \mathbb{P}(A \le s))(\mathbb{1}_{U\le t} - \mathbb{P}(U \le t))(\mathbb{1}_{V\le u} - \mathbb{P}(V \le u))) \big| .$$

Since f is monotonic, we infer that, for all $j > i > 0$,

$$\bar{\alpha}(A, X_i, X_j) \leq \bar{\alpha}(A, Y_i, Y_j) \leq \alpha_{2,\mathbf{Y}}(i)/2\,.$$

and (5.14) follows. From the previous upper bounds for Q_f and $\alpha_{2,\mathbf{Y}}(k)$, we obtain that, for $j > i > 0$,

$$\|\mathbb{E}(X_iX_j|Y_0) - \mathbb{E}(X_iX_j)\|_1 \leq \frac{C}{i}\,,$$

and (5.4) follows easily from (5.9) and (5.10).

5.3 Proof of Proposition 4.2

Let $v_0 : [0,1] \to [0,1/2]$ and $v_1 : (0,1] \to (1/2,1]$ be the two inverse branches of T_γ. Let $x_0 = 1$, and $x_n = v_0(x_{n-1})$. Let $I_n = (x_{n+1}, x_n]$, so that T_γ^n is bijective from I_n to $I_0 = (1/2,1]$. Let also h be the density of ν_γ with respect to the Lebesgue measure λ on $[0,1]$.

We use the decomposition given in Dedecker *et al.* (2008):

$$K_\gamma^n f = \sum_{i+j+k=n} A_i T_j B_k f + C_n f\,, \tag{5.15}$$

where the operators A_n, B_n and C_n are defined as follows:

$$A_n f(x) = \mathbb{1}_{[0,1/2]}(x)\frac{(v_1 v_0^{n-1})'(x) h(v_1 v_0^{n-1} x)}{h(x)} f(v_1 v_0^{n-1} x)\,, \tag{5.16}$$

$$B_n f(x) = \mathbb{1}_{(1/2,1]}(x)\frac{(v_0^n)'(x) h(v_0^n x)}{h(x)} f(v_0^n x)\,, \tag{5.17}$$

$$C_n f(x) = \mathbb{1}_{[0,1/2]}(x)\frac{(v_0^n)'(x) h(v_0^n x)}{h(x)} f(v_0^n x)\,. \tag{5.18}$$

The operator T_n is less explicit, but it can handled as follows. Let $H_\delta([a,b])$ be the space of δ-Hölder functions on $[a,b]$ equipped with the norm $|f|_{\delta,[a,b]} = \mathcal{L}_{\delta,[a,b]}(f) + \|f\|_\infty$, where

$$\mathcal{L}_{\delta,[a,b]}(f) = \sup_{x,y\in[a,b]} \frac{|f(x)-f(y)|}{|x-y|^\delta}\,.$$

According to Section 3 in Gouëzel (2007) and to Section 6.3 in Gouëzel (2004b), we have that

$$T_n = \sum_{\ell=1}^{\infty} \sum_{k_1+\cdots+k_\ell=n} R_{k_1}\ldots R_{k_\ell}\,,$$

where $(R_n)_{n\geq 1}$ is a sequence of continuous linear operators on $H_\delta([1/2,1])$ such that

$$|R_n(f)|_{\delta,[1/2,1]} \leq C\frac{|f|_{\delta,[1/2,1]}}{n^{1/\gamma+1}}\,.$$

Consequently, we can apply Theorem 2.4.10 and Remark 2.4.11 in Gouëzel (2004c) to derive that

$$|(T_n - PT_nP)(f)|_{\delta,[1/2,1]} \leq C\frac{|f|_{\delta,[1/2,1]}}{n^{1/\gamma}}\,, \tag{5.19}$$

where

$$P(f) = \frac{\nu_\gamma(f\mathbb{1}_{[1/2,1]})}{\nu_\gamma([1/2,1])}\mathbb{1}_{[1/2,1]}\,. \tag{5.20}$$

We proceed now as in the proof of Theorem 2.4.13 in Gouëzel (2004c). Let $Z_j = PT_jP$ and $Y_j = T_j - Z_j$, so that

$$|Y_j(f)|_{\delta,[1/2,1]} \leq C\frac{|f|_{\delta,[1/2,1]}}{j^{1/\gamma}}\,. \tag{5.21}$$

Notice that

$$Z_j(f) = z_j\nu_\gamma(f\mathbb{1}_{]1/2,1]})\mathbb{1}_{]1/2,1]},\ \text{where } z_j = \frac{\nu_\gamma(T_j(\mathbb{1}_{]1/2,1]})\mathbb{1}_{]1/2,1]}}{\nu_\gamma^2(]1/2,1])}\,.$$

Setting $\lambda_k(f) = \nu_\gamma(B_k(f))$, we have the following decomposition

$$K_\gamma^n(f) = C_n(f) + \sum_{i+j+k=n} z_j\lambda_k(f)A_i(\mathbb{1}_{]1/2,1]}) + \sum_{i+j+k=n} A_iY_jB_k(f)\,. \tag{5.22}$$

We shall prove successively that

$$\begin{aligned}
\|C_n(f)\|_\infty &\leq C(f)/(n+1)^{a/\gamma} \text{ for all } f \text{ in } \mathcal{H}_{0,\gamma,a} \text{ with } \nu_\gamma(f)=0\,, && (5.23)\\
\|A_n(u)\|_\infty &\leq K\|u\|_\infty/(n+1) \text{ for all bounded function } u\,, && (5.24)\\
|B_n(u)|_{\delta,[1/2,1]} &\leq C|u|_{\delta,[0,1]}/(n+1)^{1/\gamma} \text{ for all } u \text{ in } H_\delta([0,1])\,, && (5.25)\\
\lambda_n(f) &\leq C(f)/(n+1)^{(a+1)/\gamma} \text{ for all } f \text{ in } \mathcal{H}_{0,\gamma,a} \text{ with } \nu_\gamma(f)=0\,. && (5.26)
\end{aligned}$$

Let us complete the proof of Proposition 4.2 with the help of these upper bounds. Clearly, it suffices to prove the result for functions f in $\mathcal{H}_{0,\gamma,a}$ such that $\nu_\gamma(f) = 0$. Using (5.24), (5.21) and (5.25), we get that

$$\sum_{i+j+k=n} \|A_iY_jB_k(f)\|_\infty \leq C|f|_{\delta,[0,1]} \sum_{i+j+k=n} \frac{1}{(i+1)(j+1)^{1/\gamma}(k+1)^{1/\gamma}} \leq D|f|_{\delta,[0,1]}\frac{1}{n}\,.$$

We follow the computations of the proof of theorem 2.4.13 in Gouëzel (2004c), with the difference that here $\alpha_i = \|A_i(\mathbb{1}_{]1/2,1]})\|_\infty = O(i^{-1})$ by using (5.24). Consequently,

$$\sum_{i+j+k=n} z_j\lambda_k(f)\|A_i(\mathbb{1}_{]1/2,1]})\|_\infty \leq C\Big(\frac{\ln n}{n^{(1+a-\gamma)/\gamma}} + \frac{1}{n}\Big)$$

The two latter upper bounds together with (5.23) end the proof of Proposition 4.2.

We turn now to the proof of (5.23), (5.24), (5.25) and (5.26). We will use the following facts, which can be found in Liverani *et al.* (1999):

1. The density h of ν_γ is non increasing with $h(1) > 0$, and $h(x) \sim Cx^{-\gamma}$ for some $C > 0$. Moreover if $x, y \in [A, 1]$ for $A > 0$, then $|h(x) - h(y)| \le DA^{-\gamma-1}|x - y|$, for some $D > 0$.

2. One has $x_n \sim C/n^{1/\gamma}$ for some $C > 0$. Moreover, $\lambda(I_n) = x_n - x_{n+1} \sim C/n^{(1+\gamma)/\gamma}$ for some $C > 0$. One has
$$h(x_n) \sim Cx_n^{-\gamma} \sim Dn\,. \tag{5.27}$$

3. There exists a constant $C > 0$ such that, for all $n \ge 0$ and $k \ge 0$, and for all $x, y \in I_{n+k}$,
$$\left|1 - \frac{(T_\gamma^n)'(x)}{(T_\gamma^n)'(y)}\right| \le C|T_\gamma^n x - T_\gamma^n y|\,. \tag{5.28}$$

   Integrating the above inequality, we obtain that
$$C^{-1}\frac{\lambda(I_k)}{\lambda(I_{n+k})} \le (T_\gamma^n)'(x) \le C\frac{\lambda(I_k)}{\lambda(I_{n+k})}\,. \tag{5.29}$$

To prove (5.23), we use the fact that
$$\sup_{x\in[0,1]} |f(v_0^n x)| = \sup_{x\in[0,x_n]} |f(x)| \le Cx_n^a \le Dn^{-a/\gamma}$$
and Lemma 3.3 in Dedecker *et al.* (2008) which gives that
$$\sup_{x\in[0,1/2]} \left|\frac{(v_0^n)'(x)h(v_0^n x)}{h(x)}\right| \le C\,.$$

To prove (5.24), we use Lemma 3.4 in Dedecker *et al.* (2008) which gives that
$$\sup_{x\in[0,1/2]} \left|\frac{(v_1 v_0^{n-1})'(x)h(v_1 v_0^{n-1}x)}{h(x)}\right| \le \frac{C}{n+1}\,.$$

To prove (5.25), it suffices to notice that on $[1/2, 1]$ the function $1/h$ is Lipschitz, the function $h(v_0^n(x))$ is bounded by $h(x_n) \le Cn$, and the function $(v_0^n)'$ is bounded by $C/n^{(1+\gamma)/\gamma}$ by applying (5.29). Moreover for x, y in $[1/2, 1]$,
$$|h(v_0^n(x)) - h(v_0^n(y))| \le Cn^{(\gamma+1)/\gamma}|v_0^n(x) - v_0^n(y)| \le D|x - y|\,,$$
and, applying (5.28),
$$|(v_0^n)'(x) - (v_0^n)'(y)| \le Cn^{-(\gamma+1)/\gamma}|x - y|\,.$$
Gathering all these upper bounds, we obtain (5.25).

To prove (5.26), write

$$\lambda_n(f) = \nu_\gamma(B_n(f)) = \int_{1/2}^{1} f(v_0^n(x))(v_0^n)'(x)h(v_0^n(x))dx = \int_{x_{n+1}}^{x_n} f(y)h(y)dy\,.$$

Using the fact that on $[x_{n+1}, x_n]$, $|f(y)| \leq Cn^{-a/\gamma}$ and $|h(y)| \leq Cn$, (5.26) follows.

5.4 Proof of Theorems 4.4 and 4.5

Recall that $(T_\gamma, T_\gamma^2, \ldots, T_\gamma^n)$ is distributed as $(Y_n, Y_{n-1}, \ldots, Y_1)$ where $(Y_i)_{i\in\mathbb{Z}}$ is a stationary Markov chain with invariant measure ν_γ and transition kernel K_γ. Let $X_n = f(Y_n) - \nu_\gamma(f)$ and $S_n = X_1 + \cdots + X_n$. Then, for any $\varepsilon > 0$,

$$\nu_\gamma\Big(\max_{1\leq k\leq n}\Big|\sum_{i=1}^{k}(f\circ T_\gamma^i - \nu_\gamma(f))\Big| \geq \varepsilon\Big) \leq \nu_\gamma\Big(2\max_{1\leq k\leq n}|S_k| \geq \varepsilon\Big)\,. \tag{5.30}$$

Hence it remains to prove the result for the sequence $(S_k)_{k\geq1}$. To prove Theorem 4.4, we apply Corollary 1 in Peligrad *et al.* (2007). We obtain that

$$\Big\|\max_{1\leq k\leq n}|S_k|\Big\|_p \leq C_p\sqrt{n}\Big(\|X_0\|_p + \sum_{k=1}^{n} k^{-1/2}\|\mathbb{E}(X_k|Y_0)\|_p\Big)\,.$$

Since $\|\mathbb{E}(X_k|Y_0)\|_p = \|K_\gamma^n(f) - \nu_\gamma(f)\|_{p,\nu_\gamma}$ the result follows from Remark 4.3. In the same way Theorem 4.5 follows from Proposition 2 in Peligrad *et al.* (2007), and the control of $\|K_\gamma^n(f) - \nu_\gamma(f)\|_{\infty,\nu_\gamma}$ given in Proposition 4.2.

5.5 Proof of Theorem 4.7

We proceed as in the proof of Theorem 3.1 keeping the same notations. From Proposition 2 in Dedecker and Merlevède (2002), the process

$$\Big\{\frac{1}{\sqrt{n}}S_{[nt]}, t\in[0,1]\Big\}$$

converges in distribution to σW, in the Skorohod topology, as soon as

$$\sum_{k\geq1}(\ln(k))^2\|\mathbb{E}(X_k|Y_0)\|_2^2 < \infty\,, \tag{5.31}$$

with $\sigma^2 = \lim_{n\to\infty} n^{-1}\mathbb{E}(S_n^2)$. Since $\|\mathbb{E}(X_k|Y_0)\|_2 = \|K_\gamma^n(f) - \nu_\gamma(f)\|_{2,\nu_\gamma}$, it follows from Remark 4.3 that (5.31) holds as soon as $0 < \gamma < a + 1/2$.

It remains to see that $\sigma^2 = \sigma^2(f)$ defined in (4.2), which is true provided that the series $\sum_{k=0}^{\infty}|\mathbb{E}(X_0X_k)|$ converges. In Section 6.2 of Dedecker and Merlevède (2002), it is proved that

(5.31) implies that

$$\sum_{i=0}^{\infty} \|P_0(X_i)\|_2 < \infty \quad \text{where } P_k(X_i) = \mathbb{E}(X_i|Y_k) - \mathbb{E}(X_i|Y_{k-1}).$$

Since $X_k = \sum_{i=-\infty}^{k} P_i(X_k)$, and since $\mathbb{E}(P_i(X_0)P_j(X_k)) = 0$ if $i \neq j$, it follows that, for $k \geq 0$,

$$|\mathbb{E}(X_0X_k)| = \Big| \sum_{i=-\infty}^{0} \mathbb{E}(P_i(X_0)P_i(X_k)) \Big| \leq \sum_{i=0}^{\infty} \|P_0(X_i)\|_2 \|P_0(X_{k+i})\|_2 \,,$$

so that

$$\sum_{k=0}^{\infty} |\mathbb{E}(X_0X_k)| \leq \Big(\sum_{i=0}^{\infty} \|P_0(X_i)\|_2 \Big)^2 < \infty \,,$$

and the result follows.

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