On the optimality of McLeish's conditions for the central limit theorem

Jérôme Dedecker[a]

[a] Laboratoire MAP5, CNRS UMR 8145, Université Paris-Descartes, Sorbonne Paris Cité, 45 rue des Saints Pères, 75270 Paris cedex 06, France.

Abstract

We construct a family of stationary ergodic sequences for which the central limit theorem (CLT) does not hold. These examples show that McLeish's conditions for the CLT are sharp in a precise sense.

Résumé

Sur l'optimalité des conditions de McLeish pour le théorème limite central Nous construisons une famille de suites strictement stationnaires et ergodiques pour lesquelles le théorème limite central n'a pas lieu. Ces exemples montrent que les conditions de McLeish pour le théorème limite central sont optimales en un sens précis.

1 Introduction

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, and let $T : \Omega \to \Omega$ be a bijective, bi-measurable transformation preserving the probability $\mathbb{P}$. We assume here that the couple $(T, \mathbb{P})$ is ergodic, meaning that any $A \in \mathcal{A}$ satisfying $T(A) = A$ has probability 0 or 1. Let $\mathcal{F}_0$ be a σ-algebra of $\mathcal{A}$ satisfying $\mathcal{F}_0 \subseteq T^{-1}(\mathcal{F}_0)$. Let X_0 be a $\mathcal{F}_0$-measurable, square integrable and centered random variable, and define the stationary sequence $(X_i)_{i\in\mathbb{Z}}$ by $X_i = X_0 \circ T^i$. Let then

$$S_n = X_1 + X_2 + \cdots + X_n\,, \quad \text{and for } t \in [0,1], \quad W_n(t) = S_{[nt]} + (nt - [nt])X_{[nt]+1}\,.$$

Note that W_n is a random variable in the space $(C([0,1]), \|\cdot\|_\infty)$ of continuous bounded functions equipped with the uniform metric.

The following weak invariance principle (WIP) is essentially due to McLeish [10], Theorem 2.5. The present form, which can be deduced from Hannan's criterion [5][6], has been stated in [2].

Theorem 1.1 *Assume that there exists a sequence $(a_n)_{n\geq 0}$ of positive numbers such that*

$$\sum_{n\geq 0}\left(\sum_{k=0}^{n} a_k\right)^{-1} < \infty \quad \textit{and} \quad \sum_{n\geq 0} a_n \|\mathbb{E}(X_n|\mathcal{F}_0)\|_2^2 < \infty\,. \tag{1}$$

Then the series $\sigma^2 = \sum_{k\in\mathbb{Z}} \mathrm{Cov}(X_0, X_k)$ converges absolutely and $n^{-1}\mathbb{E}(S_n^2)$ converges to σ^2. Moreover the process $n^{-1/2}W_n$ converges in distribution in $(C([0,1]), \|\cdot\|_\infty)$ to σW, where W is a standard Wiener process.

Remark 1 In [2], the ergodicity is not required, and Condition (1) is shown to be sufficient for the conditional WIP (which implies the stable convergence in the sense of Rényi [12]). In [10] the non adapted case is also considered (i.e. X_0 is not supposed to be $\mathcal{F}_0$-measurable). The non-adapted version of Theorem 1.1 is given in [3]. As quoted in [3], Condition (1) holds as soon as

$$\sum_{n\geq 1} \frac{\|\mathbb{E}(X_n|\mathcal{F}_0)\|_2}{\sqrt{n}} < \infty\,. \tag{2}$$

Note that the CLT for $n^{-1/2}S_n$ under (1) was already quoted in Annexe A.2 of my PhD thesis [1] (as a consequence of Heyde's criterion [7]). At that time, my advisor Emmanuel Rio asked me whether the CLT could be true if $\sum_{n\geq 0} \|\mathbb{E}(X_n|\mathcal{F}_0)\|_2^2 < \infty$ and $n^{-1}\mathbb{E}(S_n^2) \to \sigma^2$. The answer is negative as I pointed out in Annexe A.3 of my PhD thesis (this is mentioned in Proposition 7 of [2] without proof).

Recently, the question was asked again by Christophe Cuny at the conference "*Des martingales aux systèmes dynamiques*" held in Marne-la-vallée (October 8-10, 2014). On this occasion, I showed him the counterexample of my PhD thesis, and he convinced me to write a note on the subject, and to give a more general statement. As we shall see in Corollary 2.4, the CLT is not true even if $a_n = \ln(n \vee e)$ in the second term of (1). I would like to thank here Christophe Cuny for his suggestion.

To be complete, note that the condition $\sum_{n\geq 0} \|\mathbb{E}(X_n|\mathcal{F}_0)\|_2^2 < \infty$ is sufficient for the CLT when $X_n = f(Y_n)$ is a function of a normal Markov chain, and $\mathcal{F}_0 = \sigma(Y_i, i \leq 0)$: this can be deduced from [4], as indicated to me by the referee. I would like to thank the referee, who also indicated to me Lemma 2.1 below, and the references [8] and [13].

2 Main result and discussion

Let us start with a preliminary remark. For any sequence $(\psi_n)_{n\geq 0}$ of positive numbers such that $\sum_{n\geq 0}(\sum_{k=0}^n \psi_k)^{-1} = \infty$, one can find a sequence $(u_n)_{n\geq 0}$ of positive numbers such that

$$\sum_{n\geq 0} u_n = \infty \quad \text{and} \quad \sum_{n\geq 0} \psi_n \left(\sum_{k=n}^{\infty} u_k^2\right) < \infty\,. \tag{3}$$

To see this, note that the second condition in (3) writes also $\sum_{n\geq 0} \left(\sum_{k=0}^n \psi_k\right) u_n^2 < \infty$, and that the following lemma holds.

Lemma 2.1 *Let $(v_n)_{n\geq 0}$ be a non-decreasing sequence of positive numbers. Then $\sum_{n\geq 0} v_n^{-1} = \infty$ if and only if there exists a sequence $(u_n)_{n\geq 0}$ of positive numbers such that $\sum_{n\geq 0} u_n = \infty$ and $\sum_{n\geq 0} v_n u_n^2 < \infty$.*

Proof of Lemma 2.1. If such a $(u_n)_{n\geq 0}$ exists, then, writing $u_n = u_n v_n^{1/2} v_n^{-1/2}$ and applying Cauchy-Schwarz's inequality, we see that $\sum_{n\geq 0} v_n^{-1} = \infty$. For the other implication, assume that $\sum_{n\geq 0} v_n^{-1} = \infty$ and let $y_n = \sum_{k=0}^n v_k^{-1} \to \infty$ and $u_n = (y_n v_n)^{-1}$. Since $(y_n)_{n\geq 0}$ and $(v_n)_{n\geq 0}$ are non-decreasing, one can see that $u_n \geq (y_n v_{n+1})^{-1} \geq \int_{y_n}^{y_{n+1}} x^{-1}dx$ and $v_{n+1}u_{n+1}^2 \leq \int_{y_n}^{y_{n+1}} x^{-2}dx$, and the results follows.

We are now in position to state the main result of this note.

Theorem 2.2 *Let $(\psi_n)_{n\geq 0}$ be a sequence of positive numbers such that: $\psi_n \geq 1$ for any nonnegative integer n, and $\sum_{n\geq 0}(\sum_{k=0}^n \psi_k)^{-1} = \infty$. For any sequence $(u_n)_{n\geq 0}$ of positive numbers satisfying (3), there exists a stationary ergodic sequence $(X_i)_{i\in\mathbb{Z}}$ of square integrable and centered random variables, such that*

$$\|\mathbb{E}(X_n|\mathcal{M}_0)\|_2^2 \leq \sum_{k=n}^{\infty} u_k^2 \quad \text{for } \mathcal{M}_0 = \sigma(X_i, i \leq 0), \text{ and} \quad \lim_{n\to\infty} \frac{1}{n}\mathbb{E}(S_n^2) = 1\,, \tag{4}$$

but the sequence $n^{-1/2}S_n$ does not converge in distribution.

Remark 2 Note that, by (3) and the first part of (4), $\sum_{n\geq 0} \psi_n \|\mathbb{E}(X_n|\mathcal{M}_0)\|_2^2 < \infty$. Hence, Theorem 2.2 shows that the conditions of Theorem 1.1 on the sequence $(a_n)_{n\geq 0}$ cannot be relaxed, if we assume moreover that $a_n \geq 1$ for any nonnegative integer n. Note that, by definition, $\mathcal{M}_0 = \sigma(X_i, i \leq 0)$ is the smallest σ-algebra such that X_0 is $\mathcal{M}_0$-measurable and $\mathcal{M}_0 \subseteq T^{-1}(\mathcal{M}_0)$.

Remark 3 One can modify the statement of Theorem 2.2 as follows: there exists a stationary ergodic sequence $(X_i)_{i\in\mathbb{Z}}$ of square integrable and centered random variables, such that the first condition of (4) holds, but the sequence $n^{-1/2}S_n$ is not stochastically bounded. It suffices to choose the sequence $\alpha_i = u_i(\mathbf{1}_{[0,1/2]} - \mathbf{1}_{]1/2,1]})$ in the construction of Section 3.

We now give some examples of weights satisfying the assumptions of Theorem 1.1 or Theorem 2.2.

Definition 2.3 *For $x > 0$, let $\ell_1(x) = \ln(x \vee e)$, and for any integer $k \geq 2$, define ℓ_k by induction as follows: for $x > 0$, $\ell_k(x) = \ell_1 \circ \ell_{k-1}(x)$. For any positive integer k and any positive number a, define then*

$$L_{1,a}(x) = (\ell_1(x))^a \quad \text{and for } k > 1, \quad L_{k,a}(x) = \left(\prod_{i=1}^{k-1} \ell_i(x)\right) (\ell_k(x))^a \,.$$

The following corollary is a direct consequence of Theorem 1.1 and Theorem 2.2.

Corollary 2.4 *For any positive integer k, the following statements hold:*

1. *If for some $k \geq 1$ and $a > 1$, $\sum_{n\geq 0} L_{k,a}(n)\|\mathbb{E}(X_n|\mathcal{F}_0)\|_2^2 < \infty$, then Condition (1) is satisfied, and the conclusion of Theorem 1.1 holds.*
2. *For any $k \geq 1$, there exists a stationary ergodic sequence $(X_i)_{i\in\mathbb{Z}}$ of square integrable and centered random variables, such that $\sum_{n\geq 0} L_{k,1}(n)\|\mathbb{E}(X_n|\mathcal{M}_0)\|_2^2 < \infty$ for $\mathcal{M}_0 = \sigma(X_i, i \leq 0)$ and $n^{-1}\mathbb{E}(S_n^2) \to 1$, but the sequence $n^{-1/2}S_n$ does not converge in distribution.*

Let us mention that the counterexample given in Theorem 2.2 is different from the counterexample of Peligrad and Utev [11]. In Theorem 1.2 of their paper, they show that, for any sequence $c_n \to 0$, there exists a stationary ergodic sequence $(X_i)_{i\in\mathbb{Z}}$ of square integrable and centered random variables (in their example $X_i = g(Y_i)$ where Y_i is a countable Markov chain and $\mathcal{F}_0 = \sigma(Y_i, i \leq 0)$) such that

$$\sum_{n=1}^{\infty} c_n \frac{\|\mathbb{E}(S_n|\mathcal{F}_0)\|_2}{n^{3/2}} < \infty \,, \tag{5}$$

but $n^{-1/2}S_n$ is not stochastically bounded. This proves that the condition of Maxwell and Woodroofe [9] (Condition (5) with $c_n \equiv 1$) for the CLT and the WIP (see again [11] for the WIP) is sharp (note that (2) also implies Maxwell-Woodroofe's condition). The counterexample of Peligrad and Utev is different from ours because firstly we deal with the quantity $\|\mathbb{E}(X_n|\mathcal{F}_0)\|_2$ instead of $\|\mathbb{E}(S_n|\mathcal{F}_0)\|_2$, and secondly in our case $n^{-1}\mathbb{E}(S_n^2) \to 1$ which implies the stochastic boundedness of $n^{-1/2}S_n$.

In the paper [13], there is an example of a stationary ergodic sequence such that $\|\mathbb{E}(S_n|\mathcal{F}_0)\|_2 = o(\sqrt{n}/\ln(n))$ and the CLT fails, but again the variance does not grow linearly. In [8] there is an example for which $\|\mathbb{E}(S_n|\mathcal{F}_0)\|_2 = o(\sqrt{n/\ln(n)})$ and $n^{-1}\mathbb{E}(S_n^2) \to 1$ and the CLT fails.

In Remark 2 of [8] the authors ask the following question: does $\|\mathbb{E}(S_n|\mathcal{F}_0)\|_2 = o(\sqrt{n}/\ln(n))$ and $n^{-1}\mathbb{E}(S_n^2) \to 1$ imply the CLT? Thanks to Theorem 2.2 we are able to give a negative answer to this question: it suffices to take $\psi_n \equiv 1$ and $u_n = (nL_{2,1}(n))^{-1}$, which implies that $\|\mathbb{E}(X_n|\mathcal{F}_0)\|_2 = o(n^{-1/2}/\ln(n))$. Note that this also proves that condition (2) is sharp.

3 The counterexamples

For $i \in \mathbb{Z}$, let $\mathcal{F}_i = T^{-i}(\mathcal{F}_0)$, $\mathcal{F}_{-\infty} = \cap_{i\in\mathbb{Z}}\mathcal{F}_i$, and for $i, k \in \mathbb{Z}$, let $P_i(X_k) = \mathbb{E}(X_k|\mathcal{F}_i) - \mathbb{E}(X_k|\mathcal{F}_{i-1})$. Clearly, if $\mathbb{E}(X_0|\mathcal{F}_{-\infty}) = 0$ almost surely, then $\mathbb{E}(X_k|\mathcal{F}_0) = \sum_{i\leq 0} P_i(X_k)$. The random variables $P_i(X_k), P_j(X_k)$ being orthogonal if $i \neq j$, Pythagoras's theorem and the stationarity imply that

$$\|\mathbb{E}(X_n|\mathcal{F}_0)\|_2^2 = \sum_{k=n}^{\infty} \|P_0(X_i)\|_2^2 \,. \tag{6}$$

The construction of the sequences $(X_i)_{i\in\mathbb{Z}}$ is based on the following lemmas (proofs in Section 4).

Lemma 3.1 *Let $T_{p,n}=\sum_{i=p}^{p+n}P_p(X_i)$. If $\sum_{n\geq 0}\|\mathbb{E}(X_n|\mathcal{F}_0)\|_2^2<\infty$, then*

$$\lim_{n\to\infty}\frac{\|\mathbb{E}(S_n|\mathcal{F}_0)\|_2}{\sqrt{n}}=0 \quad \textit{and} \quad \lim_{n\to\infty}\frac{1}{\sqrt{n}}\left\|S_n-\sum_{p=1}^{n}T_{p,n}\right\|_2=0\,. \tag{7}$$

Lemma 3.2 *Let $(H,\|\cdot\|_H)$ be an infinite dimensional Hilbert space. Let $(u_n)_{n\geq 0}$ be a sequence of positive numbers such that $\sum_{n\geq 0}u_n=\infty$ and $u_n\to 0$. There exists an increasing sequence of positive integers $(t_i)_{i\geq 0}$ such that: for any orthonormal family $(e_i)_{i\geq 0}$ in H, there exists a sequence $(h_i)_{i\geq 0}$ satisfying*

$$h_{t_i}=e_i,\quad \|h_j-h_{j-1}\|_H^2\leq u_j^2 \textit{ for any } j\geq 1, \tag{8}$$

and for $t_i<j<t_{i+1}$, $h_j=b_je_i+c_je_{i+1}$ with $b_j^2+c_j^2=1$.

Let us now construct the sequence $(X_i)_{i\in\mathbb{Z}}$ of Theorem 2.2. Let λ be the Lebesgue measure over $[0,1]$, and let $\mathcal{B}$ be the σ-algebra of Borel sets of $[0,1]$. We denote by $\mathcal{X}$ the probability space $\mathcal{X}=([0,1],\mathcal{B},\lambda)$. Let $(\psi_n)_{n\geq 0}$ and $(u_n)_{n\geq 0}$ be two sequences as in Theorem 2.2. Let $(\alpha_i)_{i\geq 0}$ be a sequence of functions in $H=\mathbb{L}^2(\mathcal{X})$ (to be chosen latter) such that

$$\lambda(\alpha_i)=0\,,\quad \left\|\sum_{i=0}^{n}\alpha_i\right\|_2^2=1 \quad\text{and}\quad \|\alpha_i\|_2^2\leq u_i^2\,. \tag{9}$$

We consider the space $\Omega=\mathcal{X}^{\otimes\mathbb{Z}}$ and the probabilty $\mathbb{P}=\lambda^{\otimes\mathbb{Z}}$. The transformation T is the shift on Ω defined by $(T(\omega))_i=\omega_{i+1}$. Clearly $\mathbb{P}$ is invariant by T and the couple $(T,\mathbb{P})$ is ergodic.

Starting from the sequence $(\alpha_i)_{i\geq 0}$ and from the projections $\pi_j(\omega)=\omega_j$, we define the sequence $(A_i)_{i\geq 0}$ of functions of $\mathbb{L}^2(\mathbb{P})$ by: $A_i=\alpha_i\circ\pi_0$. The sequence $(X_i)_{i\in\mathbb{Z}}$ is then defined by:

$$X_0=\sum_{j=0}^{\infty}A_j\circ T^{-j}=\sum_{j=0}^{\infty}\alpha_j\circ\pi_{-j}\quad\text{and}\quad X_i=X_0\circ T^i=\sum_{j=0}^{\infty}A_j\circ T^{i-j}\,. \tag{10}$$

Note that these series are well defined in $\mathbb{L}^2(\mathbb{P})$ because $(A_j\circ T^{i-j})_{j\geq 0}$ is a sequence of independent random variables and $\sum_{j\geq 0}\|A_j\circ T^{i-j}\|_2^2=\sum_{j\geq 0}\|\alpha_j\|_2^2\leq\sum_{j\geq 0}u_j^2<\infty$.

Let $\mathcal{F}_i=\sigma(\pi_j,j\leq i)$. Clearly, X_0 is $\mathcal{F}_0$-measurable and $\mathcal{F}_{-\infty}$ is $\mathbb{P}$-trivial by the $0-1$ law. With the notations of the beginning of this section, we have $P_0(X_i)=A_i$ for any positive integer i. Hence, it follows from (6) and (9) that $\|\mathbb{E}(X_n|\mathcal{F}_0)\|_2^2\leq\sum_{k\geq n}u_k^2$. Note that this is also true with the σ-algebra $\mathcal{M}_0=\sigma(X_i,i\leq 0)$, and the first condition of (4) is satisfied. On the other hand $T_{p,n}=\sum_{i=p}^{p+n}P_p(X_i)=\sum_{i=0}^{n}A_i\circ T^p$. Hence, the sequence $(T_{p,n})_{1\leq p\leq n}$ is i.i.d. and

$$\left\|\frac{1}{\sqrt{n}}\sum_{p=1}^{n}T_{p,n}\right\|_2^2=\|T_{0,n}\|_2^2=\left\|\sum_{i=0}^{n}\alpha_i\right\|_2^2=1\,, \tag{11}$$

where the last equality follows from (9). Applying Lemma 3.1, we infer that $\lim_{n\to\infty}n^{-1}\mathbb{E}(S_n^2)=1$, and (4) is fully satisfied.

It remains to choose α_i in such a way that $n^{-1/2}S_n$ does not converge in distribution. To do this, we use Lemma 3.2 with an appropriate orthonormal family $(e_i)_{i\geq 0}$ of $\mathbb{L}^2(\mathcal{X})$. Let first

$$f_n=\sqrt{2^n}(\mathbf{1}_{[0,2^{-n-1}[}-\mathbf{1}_{[2^{-n-1},2^{-n}[})\quad\text{and}\quad g_n=\sum_{k=0}^{2^n-1}(\mathbf{1}_{[2k/2^{n+1},(2k+1)/2^{n+1}[}-\mathbf{1}_{[(2k+1)/2^{n+1},(2k+2)/2^{n+1}[})\,.$$

Let $(t_i)_{i\geq 0}$ be the sequence of Lemma 3.2, and define $e_{2i} = f_{t_{2i}}$ and $e_{2i+1} = g_{t_{2i+1}}$. Now, we put $\alpha_0 = h_0$ and for $i > 0$, $\alpha_i = h_i - h_{i-1}$, where $(h_i)_{i\geq 0}$ is the sequence of Lemma 3.2. Applying Lemma 3.2, we see that $(\alpha_i)_{i\geq 0}$ satisfies (9). By construction $\sum_{i=0}^{t_{2n}} \alpha_i = f_{t_{2n}}$ and $\sum_{i=0}^{t_{2n+1}} \alpha_i = g_{t_{2n+1}}$.

Let us check that the two sequences $t_{2n}^{-1/2} S_{t_{2n}}$ and $t_{2n+1}^{-1/2} S_{t_{2n+1}}$ converge in distribution to two distinct laws. By Lemma 3.1 it is equivalent to consider the two sequences $t_{2n}^{-1/2} \sum_{p=1}^{t_{2n}} T_{p,t_{2n}}$ and $t_{2n+1}^{-1/2} \sum_{p=1}^{t_{2n+1}} T_{p,t_{2n+1}}$. Now

$$\mathbb{E}\left(\exp\left(\frac{ix}{\sqrt{t_{2n}}}\sum_{p=1}^{t_{2n}} T_{p,t_{2n}}\right)\right) = \left(\lambda\left(\exp\left(\frac{ixf_{t_{2n}}}{\sqrt{t_{2n}}}\right)\right)\right)^{t_{2n}} = \left(1 - \frac{1}{2^{t_{2n}}}\left(1 - \cos\left(x\sqrt{\frac{2^{t_{2n}}}{t_{2n}}}\right)\right)\right)^{t_{2n}}.$$

Consequently $\lim_{n\to\infty} \mathbb{E}(\exp(ixt_{2n}^{-1/2} S_{t_{2n}})) = 1$, proving that the sequence $t_{2n}^{-1/2} S_{t_{2n}}$ converges in distribution to the Dirac mass at 0. On the other hand, the random variables $(T_{i,t_{2n+1}})_{1\leq i\leq t_{2n+1}}$ are independent centered Rademacher random variables, so that $t_{2n+1}^{-1/2} \sum_{p=1}^{t_{2n+1}} T_{p,t_{2n+1}}$ and $t_{2n+1}^{-1/2} S_{t_{2n+1}}$ converges in distribution to a standard normal. As a conclusion, the sequence $n^{-1/2} S_n$ does not converge in distribution.

4 Proofs of Lemmas 3.1 and 3.2

Proof of Lemma 3.1. We begin with the first part of (7). Since for any positive integer m, the sequence $n^{-1/2}\|\mathbb{E}(S_m|\mathcal{F}_0)\|_2$ converges to 0 as n tends to infinity, it suffices to prove that

$$\lim_{m\to\infty} \sup_{n>m} \frac{1}{\sqrt{n}} \|\mathbb{E}(S_n - S_m|\mathcal{F}_0)\|_2 = 0\,. \tag{12}$$

Now $\|\mathbb{E}(S_n - S_m|\mathcal{F}_0)\|_2^2 = \sum_{i=m}^n \sum_{j=m}^n \mathbb{E}\left(\mathbb{E}(X_i|\mathcal{F}_0)\mathbb{E}(X_j|\mathcal{F}_0)\right) \leq 2n \sum_{i=m}^n \|\mathbb{E}(X_i|\mathcal{F}_0)\|_2^2$, where the last bound holds because $|\mathbb{E}\left(\mathbb{E}(X_i|\mathcal{F}_0)\mathbb{E}(X_j|\mathcal{F}_0)\right)| \leq \|\mathbb{E}(X_i|\mathcal{F}_0)\|_2\|\mathbb{E}(X_j|\mathcal{F}_0)\|_2 \leq \|\mathbb{E}(X_i|\mathcal{F}_0)\|_2^2$ as soon as $j \geq i$. Hence, (12) follows easily from the fact that $\sum_{n\geq 0} \|\mathbb{E}(X_n|\mathcal{F}_0)\|_2^2 < \infty$.

We now prove the second part of (7). Let $S_{p,n} = \sum_{i=p}^n P_p(X_i)$. By orthogonality and stationarity, one successively derives

$$\begin{aligned}
\frac{1}{n}\left\|S_n - \mathbb{E}(S_n|\mathcal{F}_0) - \sum_{p=1}^n T_{p,n}\right\|_2^2 &= \frac{1}{n}\left\|\sum_{p=1}^n S_{p,n} - \sum_{p=1}^n T_{p,n}\right\|_2^2 = \frac{1}{n}\sum_{p=1}^n \left\|\sum_{i=n+1}^{p+n} P_p(X_i)\right\|_2^2 \\
&= \frac{1}{n}\sum_{p=1}^n \left\|\sum_{i=n+1-p}^{n} P_0(X_i)\right\|_2^2 = \frac{1}{n}\sum_{p=1}^n \left\|\sum_{i=p}^{n} P_0(X_i)\right\|_2^2 .
\end{aligned} \tag{13}$$

Let $\beta_p = \sum_{i=p}^\infty i\|P_0(X_i)\|_2^2$ (which is finite because of (6) and the fact that $\sum_{n\geq 0} \|\mathbb{E}(X_n|\mathcal{F}_0)\|_2^2 < \infty$). Using Cauchy-Schwarz's inequality in ℓ^2, we get

$$\frac{1}{n}\sum_{p=1}^n \left\|\sum_{i=p}^{n} P_0(X_i)\right\|_2^2 \leq \frac{1}{n}\sum_{p=1}^n \beta_p \left(\sum_{i=p}^n \frac{1}{i}\right) = \frac{1}{n}\sum_{i=1}^n \left(\frac{1}{i}\sum_{p=1}^i \beta_p\right),$$

and the last term converges to zero as $n \to \infty$, by using Cesàro's lemma and the fact that $\beta_n \to 0$. Together with (13) and the first part of (7), this completes the proof of the second part of (7).

Proof of Lemma 3.2. Let $(u_n)_{n\geq 0}$ be as in Lemma 3.2 (without loss of generality, assume that $u_n < \pi/2$ for any positive integer n). Define then the increasing sequence $(t_i)_{i\geq 0}$ by induction :

$$t_0 = 0, \text{ and } t_{i+1} \text{ is the unique } n > t_i + 1 \text{ such that } \sum_{k=t_i+1}^{n-1} u_k < \frac{\pi}{2} \leq \sum_{k=t_i+1}^{n} u_k \,.$$

The function h_j is then defined by $h_{t_i} = e_i$ and, for $t_i < j < t_{i+1}$,

$$h_j = \cos\left(\sum_{k=t_i+1}^{j} u_k\right) e_i + \sin\left(\sum_{k=t_i+1}^{j} u_k\right) e_{i+1} \,.$$

By construction $\langle h_j, h_{j-1}\rangle \geq \cos(u_j)$. Hence $\|h_j - h_{j-1}\|_H^2 \leq 2(1-\cos(u_j)) \leq u_j^2$.

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