‘A mathematical overview of canonical and covariant loop quantum gravity’

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Introduction

In this thesis I collect some of the most important mathematical discoveries of a recent quantization algorithm for gauge theories and, in particular, for gravity. My original contributions in both the canonical and the covariant formulations will be underlined throughout the manuscript.

Starting from a reformulation of general relativity as a constrained gauge theory with symmetry group $SU(2)$ or $SL(2,\mathbb{C})$ − depending on which kind of signature, respectively, Euclidean or Lorentzian, is chosen to formulate the theory − then one operates a compactification of the classical configuration space $\mathcal{A}/\mathcal{G}$ of gauge field theories in order to get a quantum configuration space $\overline{\mathcal{A}/\mathcal{G}}$ that can be embedded with a natural diffeomorphism and gauge-invariant measure μ_0.

The Hilbert space $L^2(\overline{\mathcal{A}/\mathcal{G}}, \mu_0)$ servers as the space of quantum kinematical states, the true (physical) states are selected by imposing operator equations that implement the classical constraints.

By using techniques of group theory it is possible to find out an orthonormal basis of this Hilbert space, this is given by the so-called *spin-network states* and it serves to define the fundamental quantum observables of area and volume operators.

The analysis of the spectra of these observables shows that the texture of the spacetime at the ultramicroscopic scale is discrete and composed of minimal quanta of area and volume, proportional to the Planck area and volume, respectively.

These results are obviously important but are obtained in the canonical formulation at 'frozen time', in fact the evolution appears in form of a very difficult constrain, the Hamiltonian constraint, that is not yet well understood.

To overcome this difficulty in the latest years an explicitly covariant formulation has been developed. This formulation mixes techniques derived from topological field theories and from canonical loop quantum gravity.

Chapter 1

Principal bundles, gauge theories and gravity

1.1 Introduction

The principal aim of this chapter is to illustrate the mathematical framework of field theories and, in particular, of gauge theories.

The natural mathematical formalism needed to describe *any* field theory in a rigorous and exhaustive fashion is that of fiber bundles. A particular subclass of fiber bundles is given by the principal fiber bundles, these ones provide the natural framework for an important subclass of field theories called gauge theories.

Since we will need the concept of manifold, to avoid a boring proliferation of the term 'smooth' we make the following

Assumption: unless otherwise indicated the **manifolds** considered in this chapter will be assumed to be **ordinary**, i.e. smooth, connected, paracompact and finite dimensional, the **maps** between them will be assumed to be **smooth**, i.e. $\mathcal{C}^\infty$.

1.2 Field theories and fiber bundles

1.2.1 Definition of a fiber bundle

The main reference of this section is [36].

By a physical point of view a **field** is an entity which assigns to every point x of a manifold M, representing space or spacetime, a point f of another manifold F, representing the value assumed by the field in x.

A **configuration** of a field on an open subset U of the spacetime M is thus a map $\varphi : U \subset M \to F$. Such a map is completely defined by its graph, i.e. by the set

$$\mathrm{Graph}(\varphi) := \{(x, f) \in U \times F \mid f = \varphi(x)\}$$

or equivalently by the map $\hat{\varphi} : U \to U \times F$, $x \mapsto (x, \varphi(x))$.

It is quite natural to think at $U \times F$ as the local model of a more complicated geometric structure obtained by 'gluing together' these cartesian products (in a suitable way). This structure is precisely what is called a fiber bundle over M with standard fiber F. Hence, naively, a fiber bundle can be seen as a generalization of the concept of a manifold, now modelled on a cartesian product instead of an Euclidean space.

Quoting [36]: *'the mathematical formulation of the field theories in terms of the fiber bundles is an empirical consequence of the physical manifestation of a field'.*

Rigorously, the definition of a fiber bundle is the following

Def. 1.2.1 *A* **fiber bundle** *is a quadruple* $\mathcal{B} \equiv (B, M, \pi; F)$ *consisting of:*

- *a manifold B, called* **total space***;*
- *a manifold M, called* **base space***;*
- *a manifold F, called* **standard fiber***;*
- *a* **surjective submersion** $\pi : B \to M$ *called* **projection** *with the property of* **local triviality***, i.e. there exists an open covering $\{U_\alpha\}$ of M and diffeomorphisms*

  $$t_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F$$

  such that the following diagram commute:

  $$\begin{array}{ccc} \pi^{-1}(U_\alpha) & \xrightarrow{t_\alpha} & U_\alpha \times F \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle pr_1} \\ U_\alpha & = & U_\alpha \end{array}$$

i.e. $pr_1 \circ t_\alpha = \pi$, $\forall \alpha$, where pr_1 is the projection onto the first Cartesian factor.

Every couple (U_α, t_α) is called a **local trivialization** *of $\mathcal{B}$ or, in the physical language, '***choice of a gauge***'. The whole collection $\{(U_\alpha, t_\alpha)\}$ is called a* **trivialization** *of $\mathcal{B}$.*

If there exists a global trivialization (M, t) then $B \simeq M \times F$ and the fiber bundle is called **trivial***.*

The diagram above is required to commute as a natural compatibility condition between the trivializing maps t_α and the projection π.

Note, from the very definition, that if one restricts the trivializing maps t_α to the inverse-image of a specific point $x \in U_\alpha$ via π, then the restricted map $t_\alpha(x) := t_\alpha|_{\pi^{-1}(x)}$,

$$t_\alpha(x) : \pi^{-1}(x) \longrightarrow F$$

is a (non-canonical) diffeomorphism onto F.

The set $P_x \equiv \pi^{-1}(x) \subset B$ is called the **fiber** over $x \in U_\alpha$ and it is given by the elements of B which project onto x via π, hence *fibers over different points are disjoint*. Moreover, since π is a surjective submersion, each fiber is a submanifold of B (this is the reason why π is required to have that property). After the choice of a trivialization of $\mathcal{B}$ every fiber can be identified with the standard fiber F (which must be thought as an intrinsically structured manifold on its own), however the identification is not canonical because it depends on the chosen trivialization.

Fixed a trivialization $\{(U_\alpha, t_\alpha)\}$ of $\mathcal{B}$ we write $U_{\alpha\beta}$ to indicate the (possibly empty) intersection $U_\alpha \cap U_\beta$.

Supposing that $U_{\alpha\beta} \neq \emptyset$, one can define on the following maps:

$$\begin{array}{rccl} g_{\alpha\beta} : & U_{\alpha\beta} & \longrightarrow & \mathrm{Diff}(F) \\ & x & \mapsto & g_{\alpha\beta}(x) := t_\alpha(x) \circ t_\beta(x)^{-1} \end{array}$$

which are called the **transition functions** of $\mathcal{B}$ w.r.t. the chosen trivialization.

By definition the maps $g_{\alpha\beta}$ satisfy the so-called **cocycle identities**:

1. $g_{\alpha\alpha}(x) = id_F$;
2. $g_{\alpha\beta}(x) = g_{\beta\alpha}(x)^{-1}$;
3. $g_{\alpha\beta}(x) \circ g_{\beta\gamma}(x) \circ g_{\gamma\alpha}(x) = id_F$;

for every α and β. Note that, by using 2., the property 3. can be written as $g_{\alpha\beta}(x) \circ g_{\beta\gamma}(x) = g_{\alpha\gamma}(x)$.

The whole collection $\{(g_{\alpha\beta}, U_{\alpha\beta})\}$ is called a **Diff(F)−valued cocycle on M**.

A fundamental result is that the knowledge of M, F and of the Diff$(F)-$valued cocycle on M is exhaustive for the knowledge of the entire structure of $\mathcal{B}$ modulo fiber bundle isomorphisms, as the following theorem states.

Theorem 1.2.1 *Let F and M be manifolds, let $\{U_\alpha\}$ be an open covering of M and finally let $\{(g_{\alpha\beta}, U_{\alpha\beta})\}$ be a Diff(F)−valued cocycle on M.*

Then there is a fiber bundle $\mathcal{B} \equiv (B, M, \pi; F)$ which has $g_{\alpha\beta}$ as transition functions, moreover this bundle is unique up to a fiber bundle isomorphism.

The proof is constructive: one defines the total space B as the quotient space of the disjoint union

$$\coprod_{\alpha} U_\alpha \times F$$

modulo the equivalence relation

$$(\alpha, x, f) \sim (\beta, x', f') := \begin{cases} x = x' \\ f = g_{\alpha\beta}(x)(f') \end{cases}$$

and the local trivializations as the natural maps

$$\begin{array}{rlcl} t_\alpha : & \pi^{-1}(U_\alpha) & \longrightarrow & U_\alpha \times F \\ & [x, f]_\alpha & \mapsto & (x, f) \end{array}$$

where $[x, f]_\alpha$ denote an equivalence class in B.

The essential uniqueness of this construction refers to the concept of 'fiber bundle isomorphism', which is defined below.

Def. 1.2.2 *Given two fiber bundles* $\mathcal{B} \equiv (B, M, \pi; F)$ *and* $\mathcal{B}' \equiv (B', M', \pi'; F')$, *a* **fiber bundle morphism** *between them is a couple* (Φ, ϕ) *of maps* $\Phi : B \to B'$, $\phi : M \to M'$ *making the following diagram commutative:*

$$\begin{array}{ccc} B & \xrightarrow{\Phi} & B' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ M & \xrightarrow[\phi]{} & M' \end{array}$$

i.e. $\pi' \circ \Phi = \phi \circ \pi$.

In particular the morphism (Φ, ϕ) *is said to be a* **fiber bundle isomorphism** *if it is invertible, i.e. if there exist* $\Phi^{-1} : B' \to B$ *and* $\phi^{-1} : M' \to M$ *such that the morphism* (Φ^{-1}, ϕ^{-1}) *is the inverse of* (Φ, ϕ).

The commutativity of the diagram is equivalent to the ask that Φ *maps the fiber* $\pi^{-1}(x)$ *in the fiber* ${\pi'}^{-1}(\phi(x))$, in fact: suppose that b belongs to the fiber on x in B, i.e. $b \in \pi^{-1}(x)$, then $\pi(b) = x$ and the commutativity of the diagram is equivalent to $\pi'(\Phi(b)) = \phi(\pi(b)) = \phi(x)$ if and only if $\Phi(b)$ belongs to the fiber over $\phi(x)$ in B'.

If the map $\phi : M \to M'$ is a diffeomorphism then the morphism (Φ, ϕ) is called **strong**. In particular, if $\mathcal{B}$ and $\mathcal{B}'$ have the same base M and $\phi \equiv id_M$ then the morphism is called **vertical** and it is written simply $\Phi : B \to B'$.

The collection of all fiber bundles and their morphisms form a category, so that the terminology and the results of the category theory can be applied to fiber bundles and fiber bundle morphisms.

Now, turning to the issue of the essential uniqueness of the fiber bundle constructed by a cocycle, notice that if two fiber bundles $\mathcal{B}$ and $\mathcal{B}'$ with the same base M and the same standard fiber F have also the same transition functions w.r.t. the same open covering, then they are globally isomorphic.

1.2.2 Sections and local sections in a fiber bundle

Def. 1.2.3 *A map $\sigma : U \to \pi^{-1}(U)$ which makes the following diagram commutative:*

$$\begin{array}{ccc} \pi^{-1}(U) & = & \pi^{-1}(U) \\ \sigma \uparrow & & \downarrow \pi \\ U & \xrightarrow[id_U]{} & U \end{array}$$

i.e. $\pi \circ \sigma = id_U$, is called a **local section** *of B. If U coincides with M then σ is a global section, or simply a section, of $\mathcal{B}$.*

Note that, by definition, for every $x \in U$ one has $\pi(\sigma(x)) = x$, hence *a section always maps a point in the base in a point in the total space belonging to the fiber over it.*

Every fiber bundle has always local sections: in fact, fixed a trivialization $\{(U_\alpha, t_\alpha)\}$, every map $\varphi : U_\alpha \to F$ induces a section σ_φ defined by:

$$\begin{array}{rccl} \sigma_\varphi : & U_\alpha & \longrightarrow & \pi^{-1}(U_\alpha) \\ & x & \mapsto & \sigma_\varphi(x) := {t_\alpha}^{-1}(x, \varphi(x)) \equiv [x, \varphi(x)]_\alpha, \end{array}$$

the map φ is called the **local representative** of the section.

The existence of global sections instead is strongly subordinated to the topology of the fiber bundle.

The set of the local sections of $\mathcal{B}$ is usually indicated by $\Gamma_U(\pi)$, while the (possibly empty) set of the global sections of $\mathcal{B}$ by $\Gamma(\mathcal{B})$.

Obviously for trivial bundles $\Gamma(\mathcal{B})$ is always non empty, moreover

$$\Gamma(\mathcal{B}) \simeq \mathcal{C}^\infty(M, F) \qquad (\mathcal{B} \text{ trivial})$$

because every (global) section is completely individuated by its (global) representative $\varphi \in \mathcal{C}^\infty(M, F)$. Thus here a change of trivialization is simply a composition of φ with a diffeomorphism of $\mathrm{Diff}(F)$.

There is a simple way to understand if two or more sections can be glued together to form a section defined on the union of their domains, in fact suppose that the local sections σ_α and σ_β are defined as

$$\sigma_\alpha(x) := [x, f(x)]_\alpha, \ \sigma_\beta(x) := [x, h(x)]_\beta$$

for two maps $f : U_\alpha \to F$ and $h : U_\beta \to F$. The only way to define a section on $U_\alpha \cup U_\beta$ from σ_α and σ_β is to put

$$\sigma : U_\alpha \cup U_\beta \to \pi^{-1}(U_\alpha \cup U_\beta),\ x \mapsto \sigma(x) = \begin{cases} \sigma_\alpha & \text{if } x \in U_\alpha; \\ \sigma_\beta & \text{if } x \in U_\beta; \end{cases}$$

and this definition is well posed only when the two sections agree on the overlap of their domains, but this happens if and only if the local representatives f and h are related by $g_{\alpha\beta}(x) \circ f = h(x)$, $\forall x \in U_{\alpha\beta}$, in fact this condition assures that the classes $[x, f(x)]_\alpha$ and $[x, h(x)]_\beta$ agree on the overlap.

Notice that, since fibers over any two different points $x, y \in U_\alpha$ are disjoint, the value taken by the local section σ_φ in x and y will be a certain 'quote' in the respective fiber, thus the collection of these 'quotes' can be thought as a surface embedded in the product manifold $U_\alpha \times F$. By a physical point of view this surface contains the values taken by the physical field φ in the points of the portion U_α of the spacetime.

1.2.3 Fiber coordinates

Given a point $b \in B$ and fixed a trivialization (U_α, t_α) such that $t_\alpha(b) = (x, f) \in U_\alpha \times F$, if the coordinates of x in the chart domain U_α are x^μ and those of f in an open neighborhood contained in F are f^i, then b can locally be written as

$$b = (x^\mu, f^i),\ \mu = 1, \dots, dim(M),\ i = 1, \dots, dim(F).$$

These coordinates are called the **fibered coordinates** of b, they depend on the local trivialization chosen and form a system of local coordinates for B.

Notice that:

- Greek indexes as $\mu, \nu, \lambda, \gamma$, etc. label coordinates in the spacetime (base) manifold M;

- Latin indexes as i, j, a, b, etc. label coordinates in standard fiber F.

As an example, the expression of a fiber bundle morphism (Φ, φ) in fiber coordinates is the following:

$$\begin{cases} x'^\mu = \phi^\mu(x); \\ f'^i = \Phi^i(x, f). \end{cases}$$

1.2.4 Affine and vector bundles

Def. 1.2.4 *A fiber bundle* $(B, M, \pi; \mathbb{A})$ *is called an* **affine bundle** *if the standard fiber* $\mathbb{A}$ *is an affine space and if there is (at least) a trivialization whose transition functions take values in the affine subgroup* $\mathrm{GA}(\mathbb{A}) \subset \mathrm{Diff}(\mathbb{A})$*.*

According to this definition it is natural to call a morphism between affine bundles an **affine morphism** if it is an affine map when it is restricted to the fibers.

Def. 1.2.5 *A fiber bundle* $(B, M, \pi; V)$ *is called a* **vector bundle** *if the standard fiber* V *is a vector space and if there is (at least) a trivialization whose transition functions take values in the linear subgroup* $\mathrm{GL}(V) \subset \mathrm{Diff}(V)$*.*

As for affine bundles, it is natural to call a morphism between vector bundles a **vector morphism** if it is a linear map when it is restricted to the fibers.

If $dim(V) = k$ then the vector bundle is said to be of **rank** k. Obviously $dim(B) = dim(M) + dim(V)$.

In the particular case in which the rank k is 1 or 2 the vector bundle is called a **line bundle** or a **plane bundle**, respectively, over M.

Theorem 1.2.2 *Affine bundles and vector bundles always admit global sections.*

Proof. We give the proof only for vector bundle, for affine bundles see [36]. The key fact to prove the existence of global sections in a vector bundle is the presence of a 'special' point in the standard fiber, i.e. the zero of the vector space. If we fix it then, in every trivializing domain, the following section

$$\sigma^0{}_\alpha : U_\alpha \to \pi^{-1}(U_\alpha), \quad x \mapsto t_\alpha{}^{-1}(x, 0)$$

is well defined. Since the transition functions are linear, they preserve the zero of each fiber, hence the local zero sections satisfy the compatibility condition and can be glued together to give a global zero section σ^0. Starting from that, one can construct infinitely many global sections simply by deforming the zero section σ^0 on a compact support by a linear isomorphism. □

The fibered coordinates of an affine or vector bundle morphism are particular simple, in fact:

- **fibered coordinates in affine bundles**: by choosing (global) Cartesian coordinates P^i on the standard fiber $\mathbb{A}$, the local expression of an affine morphism is

$$\begin{cases} x'^\mu = \phi^\mu(x) \\ P'^i = A^i_j(x)P^j + T^i(x) \qquad \text{(affine transformation)}; \end{cases}$$

- **fibered coordinates in vector bundles**: by choosing Cartesian coordinates v^i on the standard fiber V, the local expression of a morphism is

$$\begin{cases} x'^\mu = \phi^\mu(x) \\ v'^i = A^i_j(x)v^j \qquad \text{(linear transformation)}. \end{cases}$$

1.3 Principal bundles

Def. 1.3.1 *A bundle $(B, M, \pi; G)$ is a* **principal bundle** *if the standard fiber is a Lie group G and there is (at least) one trivialization whose transition functions act on G by left translation $L_g \in \mathrm{Diff}(G), L_g(h) := gh$.*

For a principal bundle the commutativity of the diagram

$$\begin{array}{ccc} \pi^{-1}(U_\alpha) & \xrightarrow{t_\alpha} & U_\alpha \times G \\ {\scriptstyle \pi}\big\downarrow & & \big\downarrow{\scriptstyle pr_1} \\ U_\alpha & = & U_\alpha \end{array}$$

implies that the trivializing diffeomorphism t_α has the form:

$$t_\alpha(p) = (\pi(p), \phi_\alpha(p)) \qquad \forall p \in \pi^{-1}(U_\alpha)$$

where

$$\phi_\alpha : \pi^{-1}(U_\alpha) \to G$$

is a G-equivariant map, i.e. : $\phi_\alpha(p.g) = \phi_\alpha(p)g$, $\forall g \in G$.

One of the most important differences between vector bundles and principal bundles is that in the former class of bundles the fibers have a canonical structure of vector spaces, while in the latter the fibers do not carry a canonical structure of Lie77 groups, but rather many non-canonical (trivializing-depending) group structures. The reason is that the transition functions of vector bundles are linear maps and hence they preserve the zero of the fiber, while the left translation is not a group homomorphism, so that there is no preferred point in any fiber which is fixed by the transition functions to be selected as identity.

The concise definition given above shows that a principal bundle is a very particular case of fiber bundle, but for the later purposes it will be useful to give another (of course, equivalent) definition, which uses explicitly the action of the group G on the total space.

Def. 1.3.2 *A principal fiber bundle is denoted usually as $P \equiv P(M, G)$ or $\pi : P \to M$ and it is a fiber bundle in which the standard fiber is a Lie groups G, called* **structure group** *from mathematicians and* **gauge group** *from physicists, which is required to act on P with a right action*

$$R : P \times G \to P, \qquad (p, g) \mapsto R(p, g) \equiv p.g$$

everywhere free *and* **transitive on the fibers** *of P.*

From the fact that R is everywhere free and transitive on the fibers it follows that, fixed an arbitrary point p_0 in an assigned fiber, for every other point p belonging to the same fiber there exists one and only one $g \in G$ such that $p = p_0.g$.

It follows immediately that the p-reduced of the right action, i.e. the map

$$R_p : G \to P, \qquad g \mapsto R_p(g) := p.g$$

is a diffeomorphism between G and the fiber over $\pi(p)$, for every p, hence **all the fibers of P are set-theoretically isomorphic to G.**

Furthermore the g-reduced of R, i.e. the map

$$R_g : P \to P, \qquad p \mapsto R_g(p) := p.g$$

is a diffeomorphism of P into itself for every $g \in G$.

The **transition functions** of $P(M, G)$ are the maps:

$$\begin{array}{rccl} g_{\alpha\beta} : & U_\alpha \cap U_\beta & \to & G \\ & \pi(p) & \mapsto & g_{\beta\alpha}(\pi(p)) := \phi_\alpha(p)(\phi_\beta(p))^{-1} \end{array}$$

$\forall p \in \pi^{-1}(U_\alpha \cap U_\beta)$.

It can be proved that a principal bundle doesn't possess globally defined sections, unless it is trivial, but it always possess **local sections**. The local sections of $P(M, G)$ w.r.t. the locally trivializing collection $\{(U_\alpha, t_\alpha)\}$ are the following maps:

$$\begin{array}{rccl} \sigma_\alpha : & U_\alpha & \to & \pi^{-1}(U_\alpha) \\ & x & \mapsto & \sigma_\alpha(x) := t_\alpha^{-1}(x, e) \end{array}$$

where e is the unit of G.

Hence to define a local section one only has to consider a local trivialization and to use the fact that the standard fiber G admits a 'special' element, i.e. the unit e, then the inverse of the reduced diffeomorphism $t_\alpha : U_\alpha \times e \longrightarrow \pi^{-1}(U_\alpha)$ is the only other thing that works. From this observation is evident that a local section individuates and it's completely individuated by a local trivialization, for this reason physicists are used to **identify a local section with the choice of a gauge**.

The concepts of transition functions and local sections will be used in section 1.6 to write down an important transformation rule.

We want to end this section with a result that will is very useful for the purposes of canonical loop quantum gravity.

Theorem 1.3.1 *Let $P(M,G)$ be a principal bundle. If $G = SU(2)$ and $dim(M) = 3$ then P is a trivial bundle.*

1.4 Associated bundles

Let $P(M,G)$ be a principal bundle, F be any manifold and $\lambda : G \times F \to F$ a left action of G on F.

With these ingredients it is possible to define a new fiber bundle with the same base space M and the same standard fiber F but whose total space, denoted by $P \times_\lambda F$, is the quotient of the product manifold $P \times F$ w.r.t. the equivalence relation:

$$(p,f) \sim (p',f') \Leftrightarrow \exists g \in G \; : \; \begin{cases} p.g = p'; \\ f = \lambda(g,f'). \end{cases}$$

The equivalence classes in $P \times_\lambda F$ will be indicated with $[p,f]_\lambda$.

If we define a projection $\pi_\lambda : P \times_\lambda F \to M$ by:

$$\begin{array}{rccc} \pi_\lambda : & P \times_\lambda F & \longrightarrow & M \\ & [p,f]_\lambda & \mapsto & \pi(p) \end{array}$$

then the quadruple $(P \times_\lambda F, M, \pi_\lambda; F)$ is a fiber bundle with standard fiber F called the **associated bundle** via λ.

Given a local trivialization of P, or equivalently a local section σ_α, one can define the corresponding local trivialization of the associated bundle $P \times_\lambda F$ as follows:

$$\begin{array}{rccc} t_\alpha^{(\lambda)} : & {\pi_\lambda}^{-1}(U_\alpha) & \longrightarrow & U_\alpha \times F \\ & [\sigma_\alpha(x).g, f]_\lambda & \mapsto & (x, \lambda(g,f). \end{array}$$

By definition of associated bundle, the correspondence between σ_α and $t_\alpha^{(\lambda)}$ is one to one.

As an (important) example of associated bundle consider the **adjoint action of G on itself**:

$$\begin{array}{rccl} Ad: & G \times G & \longrightarrow & G \\ & (g,h) & \mapsto & Ad_g(h) := ghg^{-1} \end{array}$$

then, following the prescription above, one can construct the fiber bundle $P \times_{Ad} G$, usually denoted by $Ad(P)$, which is *not*, in general, a principal bundle.

Remember now that a linear representation of the structure group G supported on a vector space V induces (in a unique way) a left action of G on V, hence we have that *every linear representation of G gives rise to a (vector) bundle associated to the principal bundle $P(M,G)$.*

A fundamental example of vector bundle associated to a principal bundle via a linear representation of the gauge group G is constructed by considering the **adjoint representation of G onto its Lie algebra $\mathfrak{g}$**, i.e.

$$\begin{array}{rccl} ad: & G & \longrightarrow & \mathrm{Aut}(\mathfrak{g}) \\ & g & \mapsto & ad\, g := (Ad_g)_* \end{array}$$

where $(Ad\, g)_*$ is the push forward of the inner automorphism of G given by Ad_g.

The vector bundle $P \times_{ad} \mathfrak{g}$ is usually denoted by $ad(P)$ and plays a fundamental role in the applications to gauge theories.

There is a very useful correspondence between certain maps on P and the sections of an associated bundle: denote with $\mathrm{Eqv}_G(P,F)$ the space of the equivariant maps from P to F, i.e.

$$\mathrm{Eqv}_G(P,F) := \{f : P \to F \mid f(p.g) = \lambda(g^{-1}, f(p)), \forall g \in G\},$$

then a function $f \in \mathrm{Eqv}_G(P,F)$ induces a section $\sigma_f \in \Gamma(P \times_\lambda F)$ defined by

$$\sigma_f(\pi(p)) := [p, f(p)]_\lambda.$$

One can easily show that the correspondence

$$\begin{array}{ccc} \mathrm{Eqv}_G(P,F) & \longrightarrow & \Gamma(P \times_\lambda F) \\ f & \mapsto & \sigma_f \end{array}$$

is one to one, in other words *there is a bijection between G-equivariant maps of a principal bundle and (global) sections of every associated vector bundle.*

1.5 Gauge transformations in a principal bundle

The most connatural transformations with the structure of a principal bundle are the gauge transformations, to define them it is necessary to start introducing the concept of **automorphism of a principal bundle**.

Def. 1.5.1 *An automorphism of* $P(M,G)$ *is a* G*-equivariant diffeomorphism* $\Phi : P \to P$*, i.e.* $\Phi(p.g) = \Phi(p).g \quad \forall\, p \in P,\ \forall g \in G$.

The automorphisms of a principal bundle form a group w.r.t. functional composition, this group is indicated by $\mathrm{Aut}(P)$.

Every $\Phi \in \mathrm{Aut}(P)$ induces in a unique way a diffeomorphism $\Psi : M \to M$ of the base space by:

$$\Psi(\pi(p)) := \pi(\Phi(p)) \qquad \forall p \in P \tag{1.1}$$

well defined because π is a surjection. The map:

$$\begin{array}{rccc} \flat : & \mathrm{Aut}(P) & \longrightarrow & \mathrm{Diff}(M) \\ & \Phi & \mapsto & \flat(\Phi) = \Psi \end{array}$$

is a group homomorphism.

Def. 1.5.2 *The* **gauge transformations** *of* $P(M,G)$ *are the vertical automorphisms of* P*, i.e. those* $\Phi \in Aut(P)$ *that induce the identity diffeomorphism on* M*.*

Directly from the definition it follows that the set of the gauge transformations of $P(M,G)$ is precisely $Ker(\flat)$, hence it is a normal subgroup of $\mathrm{Aut}(P)$. Such a group is usually denoted by $\mathrm{Gau}(P)$ or by $\mathcal{G}$ and for trivial bundles it can be shown to agree with the space of smooth functions from M to G:

$$\mathcal{G} \simeq \mathcal{C}^\infty(M,G) \qquad P \text{ trivial.}$$

Geometrically *the gauge transformations* are often described by saying that they *leave untouched the point on the base space and move the elements of the fiber over that point.*

1.6 Connections in a principal bundle

The tangent bundle $T(P)$ of the total space P of a principal fiber bundle always possess, in a natural way, a sub-bundle indicated with $\mathrm{Ver}(P)$ and called **vertical sub-bundle**, whose fiber in the generic point $p \in P$ is:

$$V_pP := T_p(P_{\pi(p)})$$

i.e. the subspace of T_pP given by the tangent vectors to the elements of the fiber to which p belongs.

However a principal bundle doesn't possess in a natural way a sub-bundle which is supplementary to the vertical one. The presence of such a sub-bundle would be very useful, because in this situation $T(P)$ would be decomposed into a direct sum of sub-bundles.

A principal connection is exactly the instrument which generates this decomposition, as one can immediately see from its formal definition.

Def. 1.6.1 *A principal connection Γ on $P(M,G)$ is a smooth G-equivariant assignment of a sub-bundle $Hor(P)$ of $T(P)$ supplementary to $Ver(P)$, i.e. this assignment satisfies:*

$$1)\ T(P) \simeq Ver(P) \oplus Hor(P);$$

$$2)\ H_{p.g}P = (R_g)_* \, H_pP \qquad \forall p \in P,\ \forall g \in G.$$

where H_pP is the fiber of $Hor(P)$ to which p belongs and $(R_g)_*$ is the push-forward of R_g.

$Hor(P)$ is called the **horizontal sub-bundle** *of $P(M,G)$.*

The request 2) is introduced to have compatibility between the vertical vs. orthogonal decomposition of $T(P)$ and the right action of G on P.

Under the topological assumption of paracompactness for the manifolds involved (as assumed since the beginning) one can prove that **every principal bundle admits a principal connection**.

Since the connections considered in the sequel will always be principal, this adjective will be omitted.

Thanks to the presence of a connection one can define the **vertical and horizontal vector fields and 1-forms on** P:

Def. 1.6.2 *Let $P(M,G)$ be a principal bundle with a fixed connection Γ. A vector field X on P is said to be vertical (resp. horizontal) if $X_p \in V_pP$ (resp. $X_p \in H_pP$), $\forall p \in P$.*

Analogously a 1-form on P is said vertical (resp. horizontal) if it takes identically zero values when it is calculated on the horizontal (resp. vertical) vector fields of P.

From the splitting induced by a fixed principal connection, it follows that every vector field X on P can be decomposed in a unique way as the orthogonal sum of its **vertical component** X^v and its **horizontal component** X^h.

The concept of connection as defined above is important to understand the geometrical consequences induced on a principal bundle by its presence, but, as will be discussed in the last section of this chapter, in the applications to gauge theories it is more useful to work with a 1-form closely related to the connection and for this reason called **connection 1-form**.

To introduce the connections 1-form it is necessary to define the **fundamental vector fields** on P.

Def. 1.6.3 *For every vector field Y on G, the vector field $\tilde{Y}$ on P defined by:*

$$\tilde{Y}_p := (R_p)_* Y$$

is called the fundamental vector field associated to Y, which, by converse, is called the **generator** *of $\tilde{Y}$.*

The fundamental vector fields are easily seen to be vertical vector fields on P, moreover the next theorem holds.

Theorem 1.6.1 *The following map between $\mathfrak{g}$, the Lie algebra of G, and the space of the vertical vector field on P,*

$$\begin{array}{ccc} \mathfrak{g} & \to & \mathit{Ver(P)} \\ Y & \mapsto & \tilde{Y} \end{array}$$

is a bijection.

Indicated with $\Lambda(P;\mathfrak{g})$ the set of all the $\mathfrak{g}$-valued 1-forms on P the definition of a connection 1-form can be stated as below.

Def. 1.6.4 *A $\mathfrak{g}$-valued 1-form $A \in \Lambda(P;\mathfrak{g})$ which has these properties:*

$$a)\ A(\tilde{Y}) = Y \qquad \forall Y \in \mathfrak{g}\,;$$

$$b)\ (R_g)^* A = Ad_{g^{-1}} A \qquad \forall g \in G\,.$$

is called a connection 1-form.

The request $a)$ means that A reproduces the generators of the fundamental vector fields, while $b)$ express the equivariance of A.

The correlation between the connections on a principal bundle and the connection 1-forms is contained in the following theorem.

Theorem 1.6.2 *There is a bijection between the set of the connections on P and the set of the connection 1-forms on P.*

Thanks to the bijection expressed in the above theorem, it is common to call simply connections the connection 1-forms, in the sequel this convention will be adopted.

The set of all connections A has the structure of an affine manifold and the typical symbol used to indicate it is $\mathcal{A}$.

The local expressions of a connection A are obtained by taking the pull-back of A w.r.t. the local sections σ_α of $P(M, G)$. This operation gives rise to $\mathfrak{g}$-valued 1-forms locally defined on M:

$$A_\alpha := {\sigma_\alpha}^* A \ \in \Lambda(U_\alpha; \mathfrak{g})$$

for every choice of gauge (U_α, ψ_α), or of a local section σ_α.

To understand the behavior of A_α under a change of gauge it is necessary to introduce a natural $\mathfrak{g}$-valued 1-form on the gauge group G:

Def. 1.6.5 *The* **Maurer-Cartan 1-form** $\Theta \in \Lambda(G; \mathfrak{g})$ *is defined by:*

$$\Theta(X) := (L_{g^{-1}})_*(X) \qquad g \in G,\ X \in T_g G$$

where $L_{g^{-1}}$ is the left translation in G by the element g^{-1}.

By taking the pull-back of Θ w.r.t. the transition functions $g_{\alpha\beta}$ of $P(M, G)$ one gets $\mathfrak{g}$-valued 1-forms locally defined on M, precisely:

$$\Theta_{\alpha\beta} := {g_{\alpha\beta}}^* \Theta \ \in \Lambda(U_\alpha \cap U_\beta; \mathfrak{g})$$

for every overlapping couple of local trivialization (U_α, ψ_α), (U_β, ψ_β).

The next theorem is one of the most important in the applications of the theory of connections to gauge theories.

Theorem 1.6.3 *The transformation law of the local 1-forms A_α on the intersection $U_\alpha \cap U_\beta$ is:*

$$A_\beta = Ad_{g_{\alpha\beta}^{-1}} A_\alpha + \Theta_{\alpha\beta}. \tag{1.2}$$

By converse, if the family $\{A_\alpha\} \subset \Lambda(U_\alpha; \mathfrak{g})$ satisfies(1.2) then it exists one and only one connection A which has A_α as local expressions, i.e.

$$A_\alpha = {\sigma_\alpha}^* A.$$

It is worth noting that (1.2) is an **affine transformation**, so that A_α doesn't define a tensorial quantity.

The (left) action of the group of the gauge transformations $\mathcal{G}$ of a fixed principal bundle on the affine manifold of the connections $\mathcal{A}$ (on the same principal bundle) is defined by:

$$\begin{array}{rcl} \mathcal{G}\times\mathcal{A} & \to & \mathcal{A} \\ (\Phi, A) & \mapsto & \Phi^* A := A' \end{array}$$

where: $\Phi^* A(X) := A(\Phi_* X)$, for every vector field X on P.

Def. 1.6.6 *A' is called the* **gauge-transformed connection** *of the connection A. Moreover A and A' are said to be* **gauge-equivalent connections***.*

The gauge-equivalence of connections is really an equivalence relation in $\mathcal{A}$ and the space of gauge-equivalent connections is indicated by $\mathcal{A}/\mathcal{G}$.

Let's now introduce the fundamental concept of curvature of a connection.

Def. 1.6.7 *Let*

$$\begin{array}{rrcl} h: & T(P) & \longrightarrow & Hor(P) \\ & X & \mapsto & h(X) := X^h \end{array}$$

be the projector operator on the subspace of horizontal vector fields on P, then the **exterior covariant exterior derivative** *of a p-form ω is the $(p+1)$-form given by $D\omega := d\omega \circ h$, i.e.*

$$D\omega(X_1, \ldots, X_{p+1}) := d\omega(X_1^h, \ldots, X_{p+1}^h),$$

being d the exterior differential and $X_1, \ldots, X_{p+1}$ any set of vector fields on P.

In particular, the exterior covariant differential of a connection A is the 2-form $F \in \Lambda^2(P, \mathfrak{g})$, called **curvature of the connection** A. It can be proved that the curvature F of a connection A can be pulled down to M obtaining a 2-form, also indicated with F, with values in the vector bundle $ad(P)$: $F \in \Lambda^2(M, ad(P))$.

The important relation between a connection and its curvature is the contained in the **Maurer-Cartan structural relation**:

$$F(X, Y) = dA(X, Y) + [A(X), A(Y)]$$

for any couple of vector fields X, Y on P. The bracket is obviously taken in $\mathfrak{g}$, the Lie algebra of G.

Finally, the exterior differential covariant differential of the curvature F of a connection is always zero by means of the **Bianchi identity**:

$$DF = 0.$$

1.6.1 $\mathcal{A}/\mathcal{G}$, gauge fixing and Gribov ambiguity

Fixed a principal bundle $P(M,G)$, the space $\mathcal{A}$ of smooth connections on P is an infinite-dimensional affine manifold with underlying vector space $\Lambda^1(M, ad(P))$, in fact, fixed a reference connection A_0, then

$$\mathcal{A} \simeq \{A_0 + \pi^* A \mid A \in \Lambda^1(M, ad(P))\}.$$

The space $\mathcal{G}$ of smooth gauge transformations instead is a Schwartz-Lie group, i.e. a Lie group modelled on a Schwartz space, whose Lie algebra consists of the compactly supported sections of the bundle $ad(P)$.

When we give to the orbit space $\mathcal{A}/\mathcal{G}$ the quotient topology, it becomes a Hausdorff topological space, but since, in general, the action of $\mathcal{G}$ on $\mathcal{A}$ is not free, $\mathcal{A}/\mathcal{G}$ fails to be a manifold.

Let us consider now the infinite-dimensional principal bundle $\mathcal{A}(\mathcal{A}/\mathcal{G}, \mathcal{G})$ with natural projection $p : \mathcal{A} \to \mathcal{G}$. The choice of a gauge in this bundle amounts to the choice of a section $\sigma : \mathcal{A}/\mathcal{G} \to \mathcal{A}$, but this is equivalent to *fix* a 'preferred' connection (or gauge potential, see next section for details) $A \in \mathcal{A}$ in each gauge-equivalence class $[A] \in \mathcal{A}/\mathcal{G}$, this is the reason why this choice is called a **gauge fixing**.

The procedure of gauge fixing is essential in certain quantization procedures, such as the Faddeev-Popov algorithm in the context of the Feynman quantization of gauge theories.

Unfortunately, there is a big problem with this procedure, in fact Gribov has shown that the bundle $\mathcal{A}(\mathcal{A}/\mathcal{G}, \mathcal{G})$ doesn't possess global sections for almost every physical relevant case, this lack of existence of global gauge fixing is commonly called the **Gribov ambiguity**.

1.7 Gauge theories

The principal bundles provide a natural mathematical setting for gauge theories, the interested reader can find a wide discussion of this in [65].

A gauge theory can be defined as a field theory whose configuration space is $\mathcal{A}/\mathcal{G}$, i.e. whose states are parameterized by gauge-equivalence classes of connections on a principal fiber bundle, i.e. by points of $\mathcal{A}/\mathcal{G}$.

The physical meaning of the objects which compose a principal bundle is the following:

- M can represent a Cauchy hypersurface embedded in the spacetime of the theory or the spacetime itself; the first choice corresponds to the **canonical formulation**, while the second choice corresponds to the **covariant formulation** of the gauge theory;

- G is the group of the **internal symmetries** of the theory;
- P is a super-imposed structure, an auxiliary space containing the fibers over the points of M (copies of G).

In 1950 Ehresman developed in [34] the theory of connections in a fiber bundle with gauge group $SU(2)$, in 1954 Yang and Mills defined in [98] the first non-Abelian gauge theory with gauge group $SU(2)$ to describe nuclear interaction (suddenly seen to be incomplete), without knowing the work of Ehresman. The deep relation between gauge theories and the theory of connections on principal bundles has not been discovered until the early seventies, for this reason both physicists and mathematicians have developed their own language and symbology. It's then worthwhile to link the mathematical objects of principal bundle to their corresponding physical entities of gauge theories:

- a principal connection ω on P is called a **gauge connection**;
- a global section $\sigma \in \Gamma(P)$ is called a (choice of a) **global gauge**, or simply a **gauge**;
- a **gauge potential** A **on** M **in the gauge** σ is the pull back connection $A := \sigma^*(\omega) \in \Lambda^1(M, \mathfrak{g})$ (remember that such an object exists if and only if the principal bundle is trivial);
- analogously, a local section $\sigma_\alpha \in \Gamma(U_\alpha, P)$ of P is called a (choice of a) **local gauge**, the pull back connection $A_\alpha := {\sigma_\alpha}^*\omega \in \Lambda^1(U_\alpha, \mathfrak{g})$ is called a **gauge potential in the local gauge** σ_α (such objects always exist!);
- the curvature 2-form $\Omega := D\omega \in \Lambda^2(P, \mathfrak{g})$ of the gauge connection ω is called the **gauge field strength on** P, analogously the associated 2-form $F \in \Lambda^2(M, ad(P))$ is called the **gauge field strength on** M corresponding to the given gauge connection. The gauge field strength is always globally defined even though, in general, there is no corresponding globally defined gauge potential on M.

The gauge potentials and field strengths acquire physical significance only after one postulates the equations to be satisfied by them. There is (at the time of writing) no natural mathematical method for assigning field equations to gauge fields, hence these equations must be postulated in order to fit the experimental results involving the fields under analysis.

The prototype of all gauge theories is the Maxwell theory of electromagnetism: it is well known that the classical electromagnetic phenomena can be described in terms of the so called **tetravector electromagnetic potential** A_μ, $\mu = 0, \ldots, 3$, which, geometrically speaking, defines a connection on a trivial principal bundle with gauge group $U(1)$ over the Minkowski space, i.e. $\mathbb{R}^4$ endowed with the metric $\eta_{\mu\nu} = diag(-1, 1, 1, 1)$.

The **field strength of the Maxwell field** is then encoded in the curvature $F = F_{\mu\nu} dx^\mu dx^\nu$ of the connection A_μ, which is the 2-form given by $F_{\mu\nu} = (DA)_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Note that the term containing the bracket is zero since $\mathfrak{u}(1)$, the Lie algebra of $U(1)$, is Abelian.

The Maxwell equations can be written in the so-called covariant (or geometric) form, i.e. :

$$\begin{cases} dF = 0 & \text{Bianchi identity} \\ \star(d \star F) = J \end{cases}$$

where J is the current, $\star F$ is called the **dual** of F and $\star$ is the **Hodge star operator**.

If (M, g) is a n-dimensional oriented pseudo-Riemannian manifold, the Hodge star operator is the unique linear operator $\star : \Lambda^p(M) \to \Lambda^{n-p}(M)$ such that $\omega \wedge \star\tilde{\omega} = g(\omega, \tilde{\omega}) d\mathrm{v}$, for all $\omega, \tilde{\omega} \in \Lambda^p(M)$, where $g(\omega, \tilde{\omega})$ is the inner product of the two p-forms and $d\mathrm{v} := \sqrt{|det(g_{\mu\nu})|} dx^1 \wedge \ldots \wedge dx^n$, is the volume form.

The crucial fact to stress is now that if A_μ satisfies Maxwell equations then even any other potential which differs from A_μ by a gauge transformation satisfies the same equations.

Hence the physically distinct electromagnetic configurations are described by gauge-equivalence classes of connections on a principal bundle with structure group $U(1)$, this makes the electromagnetism an Abelian gauge theory.

In their work of 1954, Yang and Mills constructed non-Abelian gauge theories by replacing $U(1)$ with the compact semisimple non-Abelian Lie group $SU(2)$. By generalization, every gauge theory constructed as a theory of connections on a principal bundle with structure group $U(N)$ or $SU(N)$ is called a **Yang-Mills theory**. As in the electromagnetic case, the gauge potentials of the Yang-Mills theories are connections, whose local expressions have components given by $A_\mu : U \subset M \to \mathfrak{g}$, $\mu = 0, \ldots, dim(M)$ (the local index α has been suppressed from A and U to get a clearer notation).

The fields equations which generalize the Maxwell equations to the non-Abelian case are **the Yang-Mills equations**:

$$\begin{cases} DF = 0 \\ \star(D \star F) = J. \end{cases}$$

The most important difference between the Abelian and non-Abelian case ([13], [14]) is that the curvature F of A has a non-linear dependence on the gauge potential itself, in fact the Maurer-Cartan structural equation in local coordinates (obtained by the commutator of the covariant derivative $D_\mu = \partial_\mu + A_\mu$, see [53]) is:

$$F_{\mu\nu} = [D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$$

where the commutator $[A_\mu, A_\nu]$ is taken in $\mathfrak{g}$ and so it doesn't vanish because of the non-Abelian nature of the gauge group.

This extra term in the curvature has a dramatic consequence on the Yang-Mills equations: due to this term these equations, unlike the Maxwell equations, are non-linear. By a physical point of view this non-linearity corresponds to the presence of *self-interactions in the physics of non-Abelian gauge theories.* This fact is, for example, responsible of the fact that photons (the quanta of the electromagnetic field) don't carry electric charge (the sources of the electromagnetic field), while the gluons (the quanta of the strong nuclear field), carry color (the sources of this field).

An important feature of gauge theories to stress is that there are quantum-like experimental evidences, as the Aharonov-Bohm effect ([21] pages 130-140), which show that there can be physically observable effects of the gauge fields even where the curvature of the connection is zero, thus the interpretation of the curvature of a connection as the strength of the gauge field is physically consistent only locally and it can't represent the global configuration of the gauge field. This is a consequence of the fact that the curvature of a connections is not a gauge-invariant quantity. For this reason we are forced to parameterize the configurations of the gauge fields with gauge-equivalence classes of connections and not with their curvatures!

Finally I'd like to remember that the gauge theories are exactly the theories contemplated in the **standard model** of the nuclear and electromagnetic interactions.

In this model the particles interact by exchanging quanta of the gauge fields representing the force which makes them interact.

Precisely:

- the *quantum chromodynamics* (QCD), which describes the strong nuclear interactions, is a quantized Yang-Mills theory with gauge group $SU(3)$;

- the electromagnetic and the weak interactions are unified in the so called *electro-weak theory*, a quantized Yang-Mills theory with gauge group $SU(2) \times U(1)$. The decoupling between the weak interactions and the electromagnetic ones is described by a mechanism called *spontaneous symmetry breaking.*

The only other force in nature, the gravity, is described by Einstein's general relativity, to which is dedicated the following section.

1.8 Ashtekar's formulation of general relativity

Einstein's general relativity is the physical theory which describes how the distribution of matter and energy curves the geometry of the spacetime in which it is immersed. The way it happens is expressed by the Einstein equations.

This equations relate the **stress-energy tensor**, a symmetric $(0,2)$ tensor ($T_{\mu\nu}$ in local coordinates) which express the flow of energy and momentum through a given point of spacetime, with the curvature of the **Levi-Civita connection** ∇ associated to the metric $g_{\mu\nu}$ of the spacetime manifold.

This curvature is expressed by means of the **Riemann tensor**, defined by: $R(X,Y)Z := (\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]})Z$, or, in local coordinates:

$$R^{\mu}_{\nu\lambda\gamma} = \partial_\nu \Gamma^{\mu}_{\lambda\gamma} - \partial_\lambda \Gamma^{\mu}_{\nu\gamma} + \Gamma^{\sigma}_{\lambda\gamma}\Gamma^{\mu}_{\nu\sigma} - \Gamma^{\sigma}_{\nu\gamma}\Gamma^{\mu}_{\lambda\sigma}$$

where the Γ's are the Christoffel symbols, related to the the partial derivatives of the coordinate of the metric in this way:

$$\Gamma^{\mu}_{\nu\lambda} = \frac{1}{2} g^{\mu\sigma}(\partial_\nu g_{\lambda\sigma} + \partial_\lambda g_{\nu\sigma} - \partial_\nu g_{\nu\lambda}).$$

The trace of this tensor gives rise to the **Ricci tensor**: $R_{\mu\nu} = R^{\sigma}_{\mu\sigma\nu}$ and the contraction of the Ricci tensor gives the **scalar curvature** $R = R^{\mu}{}_{\mu}$.

This objects appear, with the metric itself, in **the Einstein equations**:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi T_{\mu\nu}$$

in units where Newton's gravitational constant is fixed to be 1.

$G_{\mu\nu}$ are the components of a symmetric $(0,2)$ tensor named **Einstein's tensor**.

Due to the symmetries of the Ricci tensor, **the Einstein equations are 10 second order hyperbolic non-linear equations in the components of the metric tensor** for every 4-dimensional spacetime.

If one imposes the Cauchy problem on these equations suddenly understand that not all of them are evolution equations, in fact 4 equations are constraints and the remaining 6 equations are evolution equations.

The reason why this happens is better understood if one consider the variational formulation of general relativity.

For the sake of simplicity, the next discussion of the actions for gravity will be focused only in the vacuum, i.e. $T_{\mu\nu} = 0$.

The first action for gravity is the Einstein-Hilbert action, i.e. a functional S on the space $Lor(M)$ of all Lorentzian metrics on a 4-D spacetime M given by:

$$S(g) := \int_M R \, d\mathrm{v}$$

where $d\mathrm{v}$ is the volume form induced by g, which can be written, in local coordinates, as $d\mathrm{v} = \sqrt{|det(g)|}dx^0 \wedge \ldots \wedge dx^3$.

The variation of S is minimized precisely when the Einstein vacuum equations hold.

The important thing to note is that this action is invariant under the action of the orientation preserving diffeomorphisms ϕ of M, i.e.:

$$\int_M (\phi^* R)\phi^* d\mathrm{v} = \int_M R \, d\mathrm{v}.$$

In general, the presence of such local symmetries implies that the Euler-Lagrange equations variationally deduced from the action (in this case the vacuum Einstein equations) are not independent and the theory, both in the Lagrangian and in the Hamiltonian formulation, is submitted to **constraints**.

Let's see what is the form of these constraints in the Hamiltonian formulation of general relativity, which is encoded in the ADM (Arnowitt-Deser-Misner) formulism.

In the ADM formulation one assumes that the spacetime M is diffeomorphic to the cartesian product $\mathbb{R} \times \Sigma$, where Σ is a 3-D spacelike slice embedded in M. This assumption is called a **splitting** of the spacetime M.

Roughly speaking, in the ADM formalism, general relativity becomes a theory which says how the curvature of Σ evolves in time.

To make this assertion rigorous one has to define the so-called **extrinsic curvature** K of Σ, which is the $(0, 2)$ symmetric tensor given by

$$K(u, v) := -g(\nabla_u v, n)$$

where u, v are tangent vectors on Σ, ∇_u is the covariant derivative defined by g, n is a unit time-like vector normal to Σ, i.e.:

$$g(n, n) = -1, \; g(n, v) = 0 \;\; \forall v \in T_p\Sigma.$$

K says how much Σ is curved in the way it sits in M, since it measures how much the unit normal vector n rotates in the direction v when it is parallel translated in the direction u.

In this formalism one can derive the so-called Gauss-Codazzi equations:

$$G_0^0 = -\frac{1}{2}({}^3R^m_{ijk} + K_{jk}K_i^m - K_{ik}K_j^m) = 0$$

$$G_i^0 = {}^3\nabla_i K_{jk} - {}^3\nabla_j K_{ik} = 0,\ i = 1, \ldots, 3$$

which says that 4 Einstein's equations are indeed constraints involving the extrinsic metric.

The objects which appear with the left suffix 3 are constructed by the **intrinsic metric** of Σ, i.e. the restriction of the metric g of M on Σ, usually written ${}^3g$.

By introducing the **shift vector field** $\vec{N}$ and the **lapse function** N one can show that the remaining 6 equations are evolutory equations which says how Σ evolves in time, in fact they contain second order time derivatives of the intrinsic curvature ${}^3g$ of Σ.

The constraints written above are very difficult to handle because they have a non-polynomial character and they are not closed under Poisson brackets.

The most important consequence of Ashtekar's formulation of general relativity [4] at a classical level is the simplification of the these constraints, which become polynomial, closed under Poisson brackets and functionally simpler. Moreover, at a quantum level, we can give a description of gravity in which these constraints are (at a kinematical level) solved, this is the celebrated **loop representation** initiated by Rovelli and Smolin [81]. The framework of loop representation of quantum gravity, or briefly '**loop quantum gravity**' will be described in wide detail in chapter 4.

Ashtekar's work is deeply related to **the Palatini formalism**, in which one consider a parallelizable oriented 4-D manifold M, i.e. it assumes that there exists a vector bundle isomorphism

$$e : \tau \equiv M \times \mathbb{R}^4 \to TM$$

inducing the identity on M. $\mathbb{R}^4$ here is called the **internal space** and capital letters $I, J, \ldots$ are used to denote its coordinates.

If $\{\xi_I\}_{I=0,\ldots,3}$ is the standard base of sections of τ then the corresponding base of vector fields on M is $\{e_I \equiv e \circ \xi_I\}_{I=0,\ldots,3}$ and e_I is locally expressed as: $e_I = e_I^\alpha \partial_\alpha$.

The Minkowski metric on each fiber defines on τ the so-called **internal metric** η.

In general the map e is called a **frame** and if the basis $\{e_I\}$ is orthonormal w.r.t. a given Lorentzian metric g on M, i.e. if $g(e_I, e_J) = \eta_{IJ}$, then the map e is called a **tetrad** or a **vierbein** for g.

Conversely, e defines a metric g on M by the formula above.

The inverse map $e^{-1} : TM \to M \times \mathbb{R}^4$ has local coordinates e_I^α satisfying $e_I^\alpha e_\alpha^J = \delta_I^J$ and is called a **cotetrad**.

The important thing to stress now is that if M is parallelizable then its principal frame bundle $\mathcal{R}M$ is also trivializable and every frame generates a trivialization by

$$\begin{array}{rcl} T: \ M \times GL(4) & \longrightarrow & \mathcal{R}M \\ (x, G) & \mapsto & T(x, G) := \{G_I^J e^I(x)\}. \end{array}$$

By considering in particular the sub-bundle $M \times SO(3,1)$ one can construct the (first order) Palatini action:

$$S(e, A) := \int_M e_I^\alpha e_J^\beta F_{\alpha\beta}^{IJ} \, d\mathrm{v}(e)$$

where e is a vierbein, A is a principal connection on $M \times SO(3,1)$, F is its curvature and $d\mathrm{v}(e)$ is the volume form defined by the Lorentzian metric g expressed as a function of e, i.e. $g_{\alpha\beta} = \eta_{IJ} e_\alpha^I e_\beta^J$.

Thus the Palatini action is a functional of a connection A and a frame e and it can be shown that varying S, with respect to both A and e, the equation $\delta S = 0$ implies that the metric $g_{\alpha\beta}$ satisfies Einstein's vacuum equations.

The Palatini formulation of general relativity has the remarkable characteristic to encode this theory in the framework of gauge theories, the price to pay is that now there are both gauge and diffeomorphism constraints, due to the invariance of $S(e, A)$ under both gauge transformations and diffeomorphisms.

Even though the form of these constraints is much simpler than in the Einstein-Hilbert approach (since they have a polynomial character), these constraints are not closed under Poisson brackets and this creates many difficulties in the canonical quantization of the theory.

Roughly speaking, Ashtekar has discovered that the Palatini action is built by using too much degrees of freedom than strictly necessary, in fact only the so-called self-dual part of F contains the geometrical information which lead to the Einstein equations. The most important consequence of the substitution of the self-dual part of F in the Palatini action is that the functional expression of the constraints simplifies and they become closed under Poisson brackets.

The starting point of Ashtekar's work is the recognition that, on the 4-dimensional Minkowski space M, the linear endomorphism given by the

Hodge star operator $* : \bigwedge^2 M \to \bigwedge^2 M$ defined on the antisymmetric $(0,2)$ tensors as:

$$*F_{IJ} = \frac{1}{2}\epsilon_{IJ}^{KL} F_{KL}$$

doesn't admit eigenvalues, but if one complexifies the theory by taking $TM \simeq M \times \mathbb{C}^4$, the Hodge star operator has eigenvalues $\pm i$ and the space $\bigwedge^2 \mathbb{C}^4$ decomposes into the direct sum of its **self-dual** and **antiself-dual** subspaces:

$$\bigwedge^2 \mathbb{C}^4 = \bigwedge^2 (\mathbb{C}^4)^+ \, \oplus \, \bigwedge^2 (\mathbb{C}^4)^-$$

which are the eigenspaces relatives to the eigenvalues $\pm i$.

The important thing to observe now is that there is an isomorphism $\bigwedge^2 \mathbb{C}^4 \simeq \mathfrak{so}(3,1) \otimes \mathbb{C}$ and, thanks to the existence of the double cover $\rho : SL(2,\mathbb{C}) \to SO_0(3,1)$, the above splitting of $\bigwedge^2(\mathbb{C}^4)$ into self-dual and antiself-dual part corresponds to the splitting of $\mathfrak{so}(3,1) \otimes \mathbb{C}$ into direct sum of $\mathfrak{sl}(2,\mathbb{C})$ with itself. The situation is clarified in this diagram:

$$\begin{array}{ccc} \bigwedge^2 \mathbb{C}^4 & \xrightarrow{\simeq} & (\bigwedge^2 \mathbb{C}^4)^+ \oplus (\bigwedge^2 \mathbb{C}^4)^- \\ \simeq \downarrow & & \downarrow \simeq \\ \mathfrak{so}(3,1) \otimes \mathbb{C} & \xrightarrow[\simeq]{} & \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C}) \end{array}$$

Since a Lorentz connection A on $M \times \mathbb{C}^4$ is just an $\mathfrak{so}(3,1) \otimes \mathbb{C}$-valued 1-form on M, the self-dual part of this connection, written usually as ${}^+A$, is a $\mathfrak{sl}(2,\mathbb{C})$-valued 1-form on M.

Thus Ashtekar modifies Palatini's formalism by introducing:

- the vector bundle $\mathbb{C}\tau = M \times \mathbb{C}^4$;
- the complexified tangent bundle $\mathbb{C}TM = \coprod_{x \in M} \mathbb{C} \otimes T_x M$;
- complex frame fields, i.e. vector bundle isomorphisms $e : \mathbb{C}\tau \to \mathbb{C}TM$;

and then defines an action, the so-called **Ashtekar's self-dual action for gravity** simply by taking the complexified Palatini action written in terms of the self-dual connection ${}^+A$ and the complex vierbein e:

$$S(e, {}^+A) := \int_M e_I^\alpha e_J^\beta {}^+F_{\alpha\beta}^{IJ} \mathrm{dv}(e)$$

quite miraculously, by varying S both w.r.t. e and ${}^+A$, one gets again the vacuum Einstein equations for the complex valued metric $g_{\alpha\beta} = \eta_{IJ} e_\alpha^I e_\beta^J$.

To obtain the usual (real) gravitation one has two following possibilities:

1. to impose reality conditions on the complex frame fields in terms of which the metric is expressed;

2. to start from an Euclidean self-dual action, defined by a volume form $d\mathrm{v}(e)$ on $\mathbb{R}^4$ induced by the (real) Riemannian metric

$$g_{\alpha\beta} = \delta_{IJ} e^I_\alpha e^J_\beta$$

and $\mathfrak{su}(2)$-valued self-dual connections ${}^+A$. By using the fact that $\mathfrak{su}(2)$ is the compact real form of $\mathfrak{sl}(2,\mathbb{C})$, one obtains again the (real) Einstein's equations. The relation between the Euclidean formulation and the Lorentzian formulation is then obtained with a generalized Wick transform, called **coherent state transform**, constructed from Ashtekar and his collaborators in [12].

The Euclidean self-dual action is invariant both under orientation preserving diffeomorphisms and under $SU(2)-$gauge transformations and, as will be explained in chapter 4, it constitutes the most important example of constrained gauge field theory to which the procedure of canonical loop quantization applies.

We want to conclude this chapter by showing which is the Hamiltonian formulation of gravity in terms of Ashtekar's variables and the corresponding, simplified, form of the constraints.

Since all $SU(2)$ bundles over three-manifolds Σ are trivial, we can fix a trivialization and regard each connection A as an $\mathfrak{su}(2)$-valued one-form on Σ. If x^a are local coordinates on Σ, $a, b, \ldots = 1, 2, 3$ and $i, j, \ldots = 1, 2, 3$ are indices of $\mathfrak{su}(2)$, the components of the connection are given by $A(x) = A(x)dx^a = A^i_a(x)\tau_i dx^a$, where $(\tau_i)_{i=1,2,3}$ is a basis of $\mathfrak{su}(2)$. The 'conjugate momenta' are non-degenerate vector densities E^a_i of weight $+1$ with values in $\mathfrak{su}(2)$. These momenta, often called *densitized triads*[1], are related to the frame fields by this formula:

$$E^a_i = \sqrt{det(q_{ab})} e^a_i$$

where q_{ab} is the inverse matrix of the contravariant, positive definite metric tensor $q^{ab} = e^a_i e^b_j k^{ij}$, being k^{ij} the Cartan-Killing tensor of $\mathfrak{su}(2)$: $k(\tau_i, \tau_j) = -2Tr(\tau_i \tau_j)$. In terms of these Riemannian structures, the area of a 2-dimensional surface S and the volume of a region R of Σ (covered, for simplicity, by a single chart) are given, respectively by:

$$A_S = \int_S \sqrt{E^a_i E^b_i n_a n_b}\, dx^1 \wedge dx^2;$$

[1]Some authors use to put a tilde sign over E to emphasize that E is a densitized object, we do not use this convention to have a clearer notation.

$$V_R := \int_R \sqrt{det(E^a_i)}\, dx^1 \wedge dx^2 \wedge dx^3;$$

where n_A is the normal to the surface S.

In the so-called triad formulation of ADM formalism for general relativity, the fields (E^a_i, K^i_a) on the slice Σ, where K^i_a is the extrinsic curvature introduced before, are assumed to be the canonical variables.

Then one jumps into Ashtekar's formalism by making a canonical transformation to a new pair of canonical variables (A^i_a, E^a_i), where $A^i_a := \Gamma^i_a + \beta K^i_a$, being Γ^i_a the spin connection of E^a_i, defined by $\partial_{[a} e^i_{b]} = \Gamma^i_{[a} e_{b]}$. β is a free factor called **Immirzi parameter**.

The Poisson bracket between A^i_a and E^a_i is the one we expect from a canonical pair:

$$\{A^i_a(x), A^j_b(y)\} = \{E^a_i(x), E^b_j(y)\} = 0,\ \ \{A^i_a(x), E^b_j(y)\} = \delta^i_j \delta^b_a \delta(x, y).$$

After this canonical transformation E^b_j re-scales of a factor $\frac{1}{\beta}$ and the constraints of general relativity in terms of the canonical pair (A, E) become

$$(1) \quad D_a E^a_i = 0;$$

$$(2) \quad E^a_i F^i_{ab} = 0;$$

$$(3) \quad \varepsilon_{ijk} E^a_i E^b_j F^k_{ab} + 2\frac{(1+\beta^2)}{\beta} E^a_i E^b_j (A^i_a - \Gamma^i_a)(A^j_b - \Gamma^j_b) = 0;$$

where D_a is the covariant derivative relative to the connection A^i_a, F^i_{ab} is the curvature of A^i_a and ε_{ijk} is the Levi-Civita density.

(1) is called '**Gauss constraint**', (2) '**diffeomorphism constraint**' and (3) '**Hamiltonian constraint**'. The polynomial nature of the first two constraints is clear, while the third is non-polynomial due to the presence of the Γ's, unless we choose to fix the Immirzi parameter β to be $-\iota$, the imaginary unit, then the second term disappears and the Hamiltonian constraint becomes polynomial too. This was Ashtekar's choice of the Immirzi parameter.

Chapter 2

Group representation theory

The technology of group representation theory will be essential to construct certain (very important) gauge-invariant states of quantum gravity (the 'spin network states'). Moreover the abstract theory of direct integral decomposition of the Hilbert space of square-integrable functions on a semisimple Lie group will be used to derive and generalize an important covariant quantization procedure (the 'Barrett-Crane model') in the last chapter.

Here we are going to introduce, in particular, the Peter-Weyl theorem, its consequences on the structure of *compact groups* and the representation theory of the non-compact group $SL(2,\mathbb{C})$.

2.1 Terminology and basic results

Given a group G, a **unitary representation** of G is given by a group homomorphism

$$\rho : G \to \mathcal{U}(V), \qquad g \mapsto \rho(g) \equiv \rho_g$$

being $\mathcal{U}(V)$ the group of the automorphisms of a Hilbert space V, which is called the **support** of ρ. The dimension of V is called the **dimension** of the representation and so, if V is not finite dimensional, we talk about *infinite dimensional representations.*

Here, unless otherwise stated, we will deal only with topological groups which are, at least, locally compact and we will consider only *continuous representations* w.r.t. the *strong topology* of the unitary operators on V, this means that the map $g \to \|\rho_g v\|$ is continuous for every $g \in G$ and every $v \in V$. We do not require the continuity w.r.t the operator norm because this would be a too restrictive condition in many situations. It is worthwhile to notice that, since the weak and strong topology coincide on $\mathcal{U}(V)$, one can equivalently ask only the weak continuity. Moreover the support space of the representations will always be a *complex* (finite or infinite dimensional) Hilbert space.

Equivalently a unitary representation of G supported on V can be defined as a left action of G on V:

$$G \times V \to V, \qquad g \mapsto g.v$$

such that the operator $\rho_g : V \to V$, $\rho_g(v) := g.v$ is a unitary operator. From the properties of an action, one has that $\rho_e = id_V$ (being e the unit of G) and $\rho_g \circ \rho_h = \rho_{gh}$, for every $g, h \in G$, so that ρ_g is actually an automorphism of V and ${\rho_g}^{-1} = \rho_{g^{-1}}$.

When V is finite dimensional, with dimension d, it is isomorphic to $\mathbb{C}^d$ and it also has the same (Euclidean) topology, by virtue of the Tychonov theorem on the equivalence of the inner products of finite dimensional vector spaces. In that case $\mathcal{U}(V)$ is the group of the unitary matrices of order d and the representation is called a **matrix representation**.

Given two representations ρ and ρ' of G, supported on V and V', respectively, an **intertwining operator**[1], or briefly an **intertwiner**, between them is a linear operator $I : V \to V'$ such that, for every $g \in G$, the following diagram is commutative:

$$\begin{array}{ccc} V & \xrightarrow{\;I\;} & V' \\ {\scriptstyle \rho_g}\big\downarrow & & \big\downarrow{\scriptstyle \rho'_g} \\ V & \xrightarrow[\;I\;]{} & V' \end{array}$$

i.e. $I \circ \rho_g = \rho'_g \circ I$. In particular, an intertwiner between a representation and itself is called an **invariant** of the representation.

If ρ and ρ' admit an intertwiner I which is also an isometric isomorphism between V and V', then they are said to be **equivalent**, and one writes $\rho \sim \rho'$. Obviously two equivalent representations have the same dimension.

If one regards the representations ρ and ρ' as actions of G on V and V', respectively, then it is obvious that the intertwining property traduces on the G-equivariance, i.e. the linear map I from V to V' is an intertwiner if and only if it satisfies the relation: $I(g.v) = g.I(v)$.

The vector space generated by the intertwiners between ρ and ρ' is indicated with $Int(V, V')$ and its dimension is called the **multiplicity** or **intertwining index** of the representation ρ supported on V into the representation ρ' supported on V' .

[1] This name is due to G.W.Mackey.

A representation ρ supported on V is called **irreducible** if there are no *proper* and *non-trivial* subspaces $E \subset V$ invariant under ρ, i.e.

$$\rho_g v \in E \qquad \forall g \in G,\ \forall v \in E$$

otherwise it is called **reducible**.

To simplify the terminology in the sequel we will call an irreducible representation simply *irrep*.

Standard results about irreps are the following.

Theorem 2.1.1 *The following facts hold:*

- **Schur's lemma***: if ρ and ρ' are irreps of G then an intertwiner between them can be only the 0 operator or an isomorphism (in this case $\rho \sim \rho'$); moreover the only invariant operators of an irrep are the multiples of the identity: $\lambda\mathbb{I}$, $\lambda \in \mathbb{C}$;*

- *all the irreps of an Abelian group are 1-dimensional;*

- *let ρ be a unitary representation of a* **compact group** *G with support on the Hilbert space V. If ρ is irreducible then V is finite dimensional.*

The theorem implies that every unitary irrep of a compact group can be implemented as a matrix representation.

The set of the equivalence classes of unitary irreps of a group G plays a fundamental role in the representation theory, it's called the **dual object** of G and it's denoted with $\hat{G}$. This space can be a group (as it happens for Abelian group) or, in general, simply a set. To denote its elements we use these notations:

- $\lambda \in \hat{G}$: equivalence class of unitary irreps of G;

- $\rho^\lambda \in \lambda$: arbitrary representative of λ.

2.2 Schur's orthogonality relations

Peter-Weyl's theory of compact groups begins with the celebrated Schur's orthogonality relations between the matrix element functions and the characters of irreps, which are defined below.

Def. 2.2.1 *Let ρ be a unitary matrix representation of dimension d of the group G. The* **matrix element functions of** ρ *are the complex-valued functions on G defined in this way:*

$$\begin{array}{rccl} \rho_{ij}: & G & \to & \mathbb{C} \\ & g & \mapsto & \rho_{ij}(g) := (\rho_g e_j \mid e_i) \end{array}$$

being $(\ \mid\ )$ the Euclidean inner product and (e_i), $i = 1, \ldots, d$, the canonical basis of $\mathbb{C}^d$.

The d^2 complex numbers $\rho_{ij}(g)$ are simply the matrix elements of position (i, j) of the unitary matrix ρ_g, for every $g \in G$.

The characters of ρ are obtained simply by saturating the matrix element functions on the same index.

Def. 2.2.2 *The* **character of the representation** ρ *is the complex-valued function on G given by:*

$$\begin{array}{rccl} \chi_\rho: & G & \to & \mathbb{C} \\ & g & \mapsto & \chi_\rho(g) := \sum_{i=1}^d \rho_{ii}(g). \end{array}$$

It's immediate to notice that, for every $g \in G$,

$$\chi_\rho(g) = Tr(\rho_g)$$

where $Tr(\rho_g)$ denotes the trace of the unitary matrix ρ_g, and so, thanks to the well known cyclic property of the trace, the characters are constant on the conjugation classes of G:

$$\chi_\rho(hgh^{-1}) = \chi_\rho(g) \qquad \forall h \in G.$$

In general, a complex-valued function which is constant on the conjugation classes of G is called a **class** (or) **central function**.

The definition of a *character of a representation* of G must not be confused with that of a *character of G*, this last one is a non-zero homomorphism from G to the group $U(1)$. The two definitions coincide if G is an Abelian group, since the irreps of an Abelian group are 1-dimensional and so the trace reduces to the identity operator.

The properties of the matrix element functions and the characters of the representations of the compact groups are very important, the first to find out these properties was Schur, in the context of the finite groups, then Burnside extended them dropping out the finiteness condition.

Before stating these properties it is worth remembering that on every locally compact group G there exists a positive regular Borel measure[2] dg, called the **Haar measure**, which is invariant under left translations, i.e. $dg(xB) = dg(B)$, for every $x \in G$ and every Borel subset B of G.

For some groups the Haar measure can be invariant under both left and right translation and under the inversion, these ones are called **unimodular groups**. It can be shown that all compact groups and all Abelian groups are unimodular, but there are many other groups which are neither compact nor Abelian (for example $SL(2, \mathbb{C})$) that share this property.

The Haar measure is unique up to multiplication by a positive real constant $c \in \mathbb{R}^+$ and it is finite if and only if G is compact. In this case one normalizes the Haar measure by fixing the multiplicative constant to be the inverse of the volume of the entire G, i.e. $c \equiv \frac{1}{dg(G)}$.

When G is finite dg is simply the counting measure of G, which assigns the number 1 to every element of G, in this case the normalization is $c = \frac{1}{|G|}$, $\mid G \mid := card(G)$, so that for a finite group the normalized Haar integrations are just means.

Thanks to the Riesz-Markov representation theorem, to dg is uniquely associated a positive linear functional I on $\mathcal{C}_c(G)$ (continuous functions on G with compact support) called the **Haar integral**:

$$I(\psi) = \int_G \psi(g)dg \qquad \psi \in \mathcal{C}_c(G),\ \ g \in G.$$

The space $L^2(G) \equiv L^2(G, dg)$ is the Hilbert space of the 'a.e.'-equivalence classes of square-Haar integrable functions $\psi : G \to \mathbb{C}$, $\int_G \mid \psi \mid^2 \, dg < \infty$ with inner product and square norm given by:

$$(\psi \mid \phi) := \int_G \psi(g)\overline{\phi(g)}dg, \qquad \|\psi\|^2 := \int_G |\psi(g)|^2 dg,$$

or

$$(\psi \mid \phi) := \frac{1}{\mid G \mid} \sum_{g \in G} \psi(g)\overline{\phi(g)}$$

for finite groups.

Let now fix our attention on the compact groups.

[2]Remember that a positive **Borel measure** $\mu : \mathcal{B}or(G) \to [0, +\infty]$ on a topological group G is a measure defined on the Borel σ-algebra $\mathcal{B}or(G)$ of G, i.e. the smallest σ-algebra generated by the open subsets of G, which assigns finite values to every compact subset of G, moreover a Borel measure is called **regular** if for every Borel subset B of G one has $\mu(B) = \sup_{K \subset B} \mu(K) = \inf_{B \subset O} \mu(O)$, where K and O are any compact and open, resp., subsets of G.

The matrix element functions and the characters of the unitary irreps of a compact group G belong to $L^2(G)$ since they are continuous and bounded functions, being the modulus of any element of a unitary matrix a finite real number. But we can say more!

Theorem 2.2.1 (Schur's orthogonality relations) *Given two unitary irreps ρ and ρ' of the compact group G, their matrix element functions ρ_{ij}, ρ'_{kl}, respectively, satisfy:*

$$(\rho_{ij} \mid {\rho'}_{kl}) = \begin{cases} \frac{1}{d}\delta_{ik}\delta_{jl} & \text{if } \rho \sim \rho' \\ 0 & \text{otherwise} \end{cases}$$

being d the (same) dimension of ρ and ρ' in case of equivalence. As a consequence their characters χ_ρ and $\chi_{\rho'}$, respectively, satisfy these **orthonormality relations***:*

$$(\chi_\rho \mid \chi_{\rho'}) = \begin{cases} 1 & \text{if } \rho \sim \rho' \\ 0 & \text{otherwise} \end{cases}$$

The orthonormality of the characters is a trivial consequence of the orthogonality of the matrix element functions, in fact

$$(\chi_\rho \mid \chi_{\rho'}) = (\sum_{i=1}^{d} \rho_{ii} \mid \sum_{k=1}^{d} {\rho'}_{kk}) = \sum_{i=1}^{d} (\rho_{ii} \mid \sum_{k=1}^{d} {\rho'}_{kk}) =$$

$$= \sum_{k=1}^{d} (\rho_{11} \mid \rho'_{kk}) + \cdots + \sum_{k=1}^{d} (\rho_{dd} \mid \rho'_{kk}) = \frac{1}{d}(\delta_{11}\delta_{11} + \cdots + \delta_{dd}\delta_{dd}) = 1$$

if $\rho \sim \rho'$ and the inner product is obviously zero otherwise.

2.3 Peter-Weyl's theorem

From the results of the previous section we see that the matrix element functions can be normalized by multiplying them by the square root of the dimension d_λ of the representations $\rho^\lambda \in \hat{G}$. What we obtain is an orthonormal system of vectors in $L^2(G)$, the most important theorem of the representation theory of compact groups, the Peter-Weyl theorem (see [73] for the very original article) states that this system is *complete*.

Theorem 2.3.1 (F.Peter-H.Weyl) *Let G be a compact group, then the family*

$$\{\sqrt{d_\lambda}\, \rho^\lambda_{ij} \mid \lambda \in \hat{G}\, ;\, i, j = 1, \ldots, d_\lambda\}$$

is a complete orthonormal system (briefly an orthonormal basis) for $L^2(G)$, called the **Peter-Weyl orthonormal basis***.*

Since the characters of the representations of G born from the matrix element functions, the Peter-Weyl theorem has consequences even on them, as stated in the next theorem.

Theorem 2.3.2 *Let G be a compact group, then:*

- *the family $\{\chi_{\rho^\lambda} \mid \lambda \in \hat{G}\}$ is an orthonormal basis for the subspace $L^2_{class}(G) \subset L^2(G)$ given by the class functions of $L^2(G)$;*
- *the dimension of ρ equals the value of χ_ρ in e, the unit element of G: $\chi_\rho(e) = d(\rho)$;*
- *a finite dimensional unitary representation ρ of G is irreducible if and only if $(\chi_\rho \mid \chi_\rho) = 1$, i.e. iff the norm of its character is 1;*
- *two unitary irreps ρ and ρ' of G are equivalent if and only if their characters coincide: $\rho \sim \rho' \Leftrightarrow \chi_\rho = \chi_{\rho'}$.*

From the general theory of Hilbert spaces we know that, given an orthonormal basis of a Hilbert space, one can always construct a Fourier series over it, let us see how to do so with the Peter-Weyl (PW from now on) orthonormal basis.

Theorem 2.3.3 *Let G be a compact group. For every $f \in L^2(G)$ the following expansion holds:*

$$f = \sum_{\lambda \in \hat{G}} d_\lambda \sum_{i,j=1}^{d_\lambda} \hat{f}^\lambda_{ij} \, \rho^\lambda_{ij}$$

where $\hat{f}^\lambda_{ij}$ are the Fourier coefficients of f w.r.t. the PW orthonormal basis, i.e.

$$\hat{f}^\lambda_{ij} := (f \mid \rho^\lambda_{ij}) = \int_G f(g)\overline{\rho^\lambda_{ij}(g)} dg \, .$$

The convergence of the Fourier series is meant in the $L^2(G)$-norm, i.e. : $\lim_{\lambda \in \hat{G}} \|f - \sum_\lambda d_\lambda \sum_{i,j=1}^{d_\lambda} \hat{f}^\lambda_{ij} \, \rho^\lambda_{ij}\|_2 = 0$.

The norm of $f \in L^2(G)$ is given by the Parseval-Plancherel identity:

$$\|f\|_2{}^2 = \sum_{\lambda \in \hat{G}} \sum_{i,j=1}^{d_\lambda} \mid \hat{f}^\lambda_{ij} \mid^2$$

notice that this formula relates a series with an integral, thus enabling to compute one by the knowledge of the other!

An analogous result holds for the functions belonging to $L^2_{class}(G)$, but now the Fourier series is over the orthonormal basis of characters of the unitary irreps of G: for every $f \in L^2_{class}(G)$ one has that

$$f = \sum_{\lambda \in \hat{G}} \hat{f}^{\lambda} \chi_{\rho^{\lambda}}$$

where $\hat{f}^{\lambda} = (f \mid \chi_{\rho^{\lambda}}) = \int_G f(g)\overline{\chi_{\rho^{\lambda}}(g)}dg$.

Let's end the list of the consequences of the Peter-Weyl theorem by showing how the Fourier transform can be extended to any compact topological group by using the PW orthonormal basis.

First of all remember that $\| \ \|_{HS}$ is the Hilbert-Schmidt norm, defined on the space of the matrices $A \in M(n, \mathbb{C})$ by[3]:

$$\|A\|^2_{HS} = \sum_{i,j=1}^{n} \mid a_{ij} \mid^2 = Tr(AA^{\dagger})$$

being $A^{\dagger} \equiv {}^t\bar{A}$.

With the dual $\hat{G}$ of the compact group G let's build the Hilbert space $L^2(\hat{G})$ given by the **operator-valued functions** $\psi : \hat{G} \to \cup_{n=1}^{\infty} M(n, \mathbb{C})$ satisfying:

- $\psi(\lambda) \in M(d_{\lambda}, \mathbb{C})$, $\forall \lambda \in \hat{G}$;
- $\sum_{\lambda \in \hat{G}} d_{\lambda} \|\psi(\lambda)\|^2_{HS} < \infty$

and endowed with the inner product

$$(\psi \mid \phi) = \sum_{\lambda \in \hat{G}} d_{\lambda}\, Tr(\psi(\lambda)\, \phi(\lambda)^{\dagger})\,.$$

The extension of the Fourier transform to the compact groups is possible by means of the following theorem.

Theorem 2.3.4 (Plancherel theorem) *The operator:*

$$\begin{array}{rccc} F: & L^2(G) & \to & L^2(\hat{G}) \\ & f & \mapsto & \hat{f} \end{array}$$

[3] To prove the equality $\sum_{i,j=1}^{n} \mid a_{ij} \mid^2 = Tr(AA^{\dagger})$ remember that the matrix element c_{ik} of the product matrix $C = AB$ it's $c_{ik} = \sum_{j=1}^{n} a_{ij}b_{jk}$, hence $Tr(C) = \sum_{i=1}^{n} c_{ii} = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}b_{ji}$, thus, if $B = A^{\dagger}$, $b_{ji} = \overline{a_{ij}}$ (transposition and conjugation) and so $Tr(AA^{\dagger}) = \sum_{i,j=1}^{n} |a_{ij}|^2$.

where $\hat{f}(\lambda) \in M(d_\lambda, \mathbb{C})$ is the matrix whose entries are the Fourier coefficients w.r.t. the PW basis, i.e.

$$\hat{f}(\lambda) := \int_G f(g)\rho^\lambda_{g^{-1}} dg = \left(\hat{f}^\lambda_{ij} := \int_G f(g)\overline{\rho^\lambda_{ij}(g)} dg \right)_{i,j=1,\dots,d_\lambda}$$

is a surjective isometry called **Fourier-Plancherel transform**.

In terms of the Fourier-Plancherel transform the expansion of an element $f \in L^2(G)$ and the Parseval-Plancherel identity can be re-written (by direct substitution) as follows:

$$f(g) = \sum_{\lambda \in \hat{G}} d_\lambda Tr(\hat{f}(\lambda)\rho^\lambda_g)$$

and

$$\|f\|_2{}^2 = \sum_{\lambda \in \hat{G}} d_\lambda \|\hat{f}(\lambda)\|^2_{HS}.$$

When applied to the torus the Peter-Weyl theory gives back the Fourier theory of harmonic oscillations, in fact if $G = U(1)$ then $\hat{G} = \mathbb{Z}$, $\rho_{g^{-1}} = \overline{\rho_g} = e^{-i\theta}$ and the Fourier-Plancherel transform reduces to the 'classical' Fourier transform (see [88] for details). This is the reason why the Peter-Weyl theory is considered the natural generalization of the Fourier theory to compact non-Abelian groups, where **the unitary irreps of compact groups generalize the role that the harmonic oscillations $e^{i\theta}$ have in the Fourier expansion of the plane waves**.

2.4 Other formulations of the Peter-Weyl theorem

The exposition of the Peter-Weyl theory given in the previous section has the advantage to put in a evidence the correlation between this theory and the Fourier one. However, for the purposes of quantum gravity, precisely for the construction of spin networks states, a more abstract (but equivalent) formulation of the Peter-Weyl theory is best suited.

The alternative formulation starts with the simple consideration that among all representations of a compact group G there are two very natural ones: the right and left regular representations.

These representations are natural in the sense that their support is the Hilbert space $L^2(G)$ and they act simply as a translation in G.

Rigorously, the **right regular representation** of G is defined by

$$\begin{array}{rccl} R: & G & \longrightarrow & \mathcal{U}(L^2(G)) \\ & g & \mapsto & R_g, \end{array}$$

$(R_g f)(h) := f(hg), \forall f \in L^2(G)$.

Analogously, the **left regular representation** of G is defined by

$$\begin{array}{rccl} L: & G & \longrightarrow & \mathcal{U}(L^2(G)) \\ & g & \mapsto & L_g, \end{array}$$

$(L_g f)(h) := f(g^{-1}h), \forall f \in L^2(G)$.

The representations R and L are unitary since the Haar measure of a compact group is invariant under right and left translations and under inversion.

The importance of the left and right regular representations is that they can be used to decompose the Hilbert space $L^2(G)$ into a direct sum of finite-dimensional subspaces which are invariant under the representations themselves.

Before showing how this decomposition can be realized let's remember the concept of direct sum and tensor product of representations.

A Hilbert space V is the direct sum of a collection $\{V_i\}$ of its subspaces if they are mutually orthogonal (i.e. $V_i \perp V_j$, whenever $i \neq j$) and if their span is dense in V (i.e. $\overline{span(\bigcup_i V_i)} = V$). If this is the case then one writes $V = \bigoplus_i V_i$.

If every V_i is **ρ-invariant** under the representation ρ of G, i.e. $v \in V_i$ implies $\rho_g(v) \in V_i$ for every $g \in G$, then ρ is said to be the **direct sum** of the sub-representations

$$\begin{array}{rccl} \rho^{(i)}: & G & \longrightarrow & \mathcal{U}(V_i) \\ & g & \mapsto & \rho_g^{(i)} := \rho_g|_{V_i} \end{array}$$

and one writes $\rho = \bigoplus_i \rho^{(i)}$.

The matrix realization of a direct sum representation is given by the block diagonal matrix in which every block represents the matrix realization of the corresponding sub-representation.

One can easily verify that if ρ is the direct sum of the representations $\rho^{(i)}$, $i = 1, \ldots, n$, then the projection operator from V to V_i is an intertwiner between ρ and $\rho^{(i)}$, for every $i = 1, \ldots, n$.

Now remember that, given two linear spaces V and V', their tensor product is isomorphic to the space of the linear operators from the dual of V' to V, i.e. $V \otimes V' := Hom((V')^*, V)$. Moreover, if $A \in End(V)$ and $B \in End(V')$, then the tensor product operator $A \otimes B$ belongs to $End(V \otimes V')$ and it is defined by:

$$\begin{array}{rcl} A \otimes B: \; V \otimes V' & \longrightarrow & V \otimes V' \\ X & \mapsto & (A \otimes B)(X) := A \circ X \circ B^{\dagger}. \end{array}$$

Given two representations ρ and ρ' of the group G, their **tensor product** $\rho \otimes \rho'$ is the representation of G supported on $V \otimes V'$ defined by:

$$\begin{array}{rcl} \rho \otimes \rho': \; G & \longrightarrow & Aut(V \otimes V') \\ g & \mapsto & (\rho \otimes \rho')(g) := \rho_g \otimes \rho'_g \end{array}$$

for every $g \in G$.

The matrix realization of a tensor product representation $\rho \otimes \rho'$ is given by the Kronecker product of the matrix corresponding to the representations ρ and ρ'. Explicitly:

$$(\rho \otimes \rho')_{ik,jl}(g) := \rho_{ij}(g)\rho'_{kl}(g)$$

where $i, j = 1, \dots, n \equiv dim(V)$ and $k, l = 1, \dots, n' \equiv dim(V')$.

It is interesting to see what is the explicit form of an intertwiner between two tensor product representations of a group: let $\rho_1, \dots, \rho_k, \dots, \rho_N$ be N representations of the group G, supported on the spaces $V_1, \dots, V_k, \dots, V_N$, respectively, then an intertwiner between $\bigotimes_{i=1}^{k} \rho_i$ and $\bigotimes_{j=k+1}^{N} \rho_j$ is a tensor

$$I \in \bigotimes_{j=k+1}^{N} V_j \otimes \bigotimes_{i=1}^{k} V_i^* \simeq Hom\left(\bigotimes_{i=1}^{k} V_i, \bigotimes_{j=k+1}^{N} V_j\right)$$

whose components $I_{n_1 \cdots n_k}^{n_{k+1} \cdots n_N}$ satisfy:

$$I_{n_1 \cdots n_k}^{n_{k+1} \cdots n_N} = \rho_{k+1}(g)_{m_{k+1}}^{n_{k+1}} \cdots \rho_N(g)_{m_N}^{n_N} \, I_{m_1 \cdots m_k}^{m_{k+1} \cdots m_N} \, \rho_1(g^{-1})_{n_1}^{m_1} \cdots \rho_k(g^{-1})_{n_k}^{m_k}$$

where the Einstein convention has been used.

Now observe that to every fixed $\lambda \in \hat{G}$ there correspond d_λ Hilbert spaces, precisely the subspaces $\mathfrak{M}_i^\lambda$ of $L^2(G)$ spanned by the rows (or, equivalently, the columns) of the matrix element functions of the irreducible representations of G, i.e.

$$\mathfrak{M}_i^\lambda := \{\sum_{j=1}^{d_\lambda} c_j \rho_{ij}^\lambda \mid c_j \in \mathbb{C}\} \qquad i = 1, \dots, d_\lambda.$$

It can be easily proven that the spaces $\mathfrak{M}_i^\lambda$ are R-invariant and the restriction of R to these subspaces coincides with ρ^λ, i.e. $R_g|_{\mathfrak{M}_i^\lambda}$ is the realized by the same matrix as ρ_g^λ, for every $\lambda \in \hat{G}$.

Now it's immediate to observe that the Peter-Weyl theorem it equivalent to say that:

$$L^2(G) \simeq \bigoplus_{\lambda \in \hat{G}} \bigoplus_{i=1}^{d_\lambda} \mathfrak{M}_i^\lambda$$

and

$$R \simeq \bigoplus_{\lambda \in \hat{G}} d_\lambda \cdot \rho^\lambda,$$

i.e. every ρ^λ appears exactly d_λ times in the direct decomposition of the right-regular representation R of G.

Since all these spaces $\mathfrak{M}_1^\lambda, \ldots, \mathfrak{M}_{d_\lambda}^\lambda$ are finite-dimensional, they are all isomorphic with $\mathbb{C}^{d_\lambda}$, for this reason the subscript i will be omitted in the sequel and the above decomposition will be shortly written as

$$L^2(G) \simeq \bigoplus_{\lambda \in \hat{G}} d_\lambda \cdot \mathfrak{M}^\lambda \ .$$

Analogous results hold for the left regular representation, i.e.

$$L^2(G) \simeq \bigoplus_{\lambda \in \hat{G}} d_\lambda \cdot \overline{\mathfrak{M}^\lambda}$$

where the Hilbert spaces $\overline{\mathfrak{M}^\lambda}$ are those spanned by the rows (or, equivalently, the columns) of the complex-conjugated of the matrix element functions of the irreducible representations of G. They are invariant under the left-regular representation L of G and this one is decomposed as

$$L \simeq \bigoplus_{\lambda \in \hat{G}} d_\lambda \cdot \overline{\rho^\lambda}.$$

For later purposes it is also useful to decompose a unitary representation of the product group $G \times G$ supported again on $L^2(G)$ that can be cooked up from the left and the right regular representations. Notice in fact that the associative group law guarantees that the operators R_g and L_h commute for every $g, h \in G$, i.e. $R_g L_h = L_h R_g$, hence the right and left regular representations can be combined to give the so-called **two-sided regular representation** defined by:

$$\begin{array}{rccl} \tau : & G \times G & \longrightarrow & \mathcal{U}(L^2(G)) \\ & (g,h) & \mapsto & \tau(g,h) := R_g L_h = L_h R_g, \end{array}$$

$(\tau(g,h)\, f)(k) := f(h^{-1}kg), \forall f \in L^2(G)$.

τ is a unitary representation of $G \times G$ thanks to the bi-invariance of the Haar measure.

The Peter-Weyl decomposition of $L^2(G)$ in terms of the representation τ is:

$$L^2(G) \simeq \bigoplus_{\lambda \in \hat{G}} \mathfrak{M}^\lambda \otimes \overline{\mathfrak{M}^\lambda};$$

$$\tau \simeq \bigoplus_{\lambda \in \hat{G}} \rho^\lambda \otimes \overline{\rho^\lambda}.$$

2.4.1 The projection on the intertwiners

The main reference of this section is [28]. Let $\rho : G \to \mathcal{U}(V)$ be a representation of the compact group G and denote with $\mathrm{Fix}_\rho(V)$ the subspace of fixed points of V under the action of $\rho(G)$, i.e.

$$\mathrm{Fix}_\rho(V) := \{v \in V \mid \rho_g(v) = v, \forall g \in G\}.$$

Then the operator

$$\begin{array}{rccl} p: & V & \longrightarrow & \mathrm{Fix}_\rho(V) \\ & v & \mapsto & p(v) := \int_G \rho_g(v) dg \end{array}$$

is a projector.

First of all let us verify that $p(v) = \int_G \rho_g(v)dg$ belongs to $\mathrm{Fix}_\rho(V)$: for every $h \in G$, $\rho_h(\int_G \rho_g(v)dg) = \int_G \rho_{hg}(v)dg$, but thanks to the invariance of the Haar measure this integral is the same as $\int_G \rho_g(v)dg$, so that $p(v)$ is unaffected by the action of G.

Moreover let's check that p is a projector, i.e. that it is idempotent: $p^2(v) = p(p(v)) = \int_G \rho_h(\int_G \rho_g(v)dg)dh = \int_G p(v)dg = p(v)\int_G dg = p(v)$, since dg is normalized to 1.

Now consider two representations $\rho : G \to \mathcal{U}(V)$ and $\rho' : G \to \mathcal{U}(V')$ of the same group G, then it's easy to construct a representation of G supported on $Hom(V, V')$ in this way:

$$\begin{array}{rccl} \eta: & G & \longrightarrow & \mathcal{U}(Hom(V,V')) \\ & g & \mapsto & \eta_g, \end{array}$$

$\eta_g(A) := \rho'_g \circ A \circ \rho_{g^{-1}} : V \to V'$, for every $A \in Hom(V, V')$. This definition is conveniently visualized with the help of the following commutative diagram:

$$\begin{array}{ccc} V & \xrightarrow{A} & V' \\ {\scriptstyle \rho_g}\downarrow & & \downarrow{\scriptstyle \rho'_g} \\ V & \xrightarrow[\eta_g(A)]{} & V' \end{array}$$

The action of G on $Hom(V,V')$ corresponding to the representation η is obviously the following:

$$\begin{array}{ccc} G \times Hom(V,V') & \longrightarrow & Hom(V,V') \\ (g,A) & \mapsto & g \circ A, \end{array}$$

$$\begin{array}{cccc} g \circ A : & V & \longrightarrow & V' \\ & v & \mapsto & (g \circ A)v := g.A(g^{-1}.v). \end{array}$$

It will be interesting to know which are the fixed points of $Hom(V,V')$ w.r.t. the representation η. First of all observe that the projector on the fixed points w.r.t. the representation η is

$$\begin{array}{cccc} p : & Hom(V,V') & \longrightarrow & \mathrm{Fix}_\eta(Hom(V,V')) \\ & A & \mapsto & p(A) := \int_G \eta_g \, dg = \int_G \rho'_g \circ A \circ \rho_g^{-1} \, dg. \end{array}$$

Moreover we can prove the following theorem.

Theorem 2.4.1 *The space of fixed points of $Hom(V,V')$ w.r.t. the representation η of G agrees with the space spanned by the intertwiners between the representations ρ and ρ', i.e.*

$$Fix_\eta(Hom(V,V')) = Int(V,V').$$

Proof.

- $\mathrm{Fix}_\eta(Hom(V,V')) \subseteq Int(V,V')$: one has to verify that

$$\left(\int_G \rho'_g \circ A \circ \rho_{g^{-1}} \, dg \right) \circ \rho_h = \rho'_h \circ \left(\int_G \rho'_g \circ A \circ \rho_{g^{-1}} \, dg \right)$$

  i.e.

$$\int_G \rho'_g \circ A \circ \rho_{g^{-1}h} \, dg = \int_G \rho'_{hg} \circ A \circ \rho_{g^{-1}} \, dg.$$

  Using again the invariance of the Haar measure, one can substitute hg to g and $g^{-1}h^{-1}$ to g^{-1} in the left hand side obtaining the same integral, i.e. $\int_G \rho'_g \circ A \circ \rho_{g^{-1}h} dg = \int_G \rho'_{hg} \circ A \circ \rho_{g^{-1}h^{-1}h} dg = \int_G \rho'_{hg} \circ A \circ \rho_{g^{-1}} dg$. This proves that the intertwining relation holds for every $A \in Hom(V,V')$.

- $Int(V,V') \subseteq \mathrm{Fix}_\eta(Hom(V,V'))$: it has to be verified that $\eta_g \circ I = I$ whenever I is an intertwining operator, but this is immediate since $\eta_g \circ I := \rho'_g \circ I \circ \rho_{g^{-1}} = I$ since I is an intertwiner.

□

In chapter 5 we will use these results to construct the spin network states of loop quantum gravity by specializing the representations ρ and ρ' to be the left and right regular representations of G and mixing the Peter-Weyl theory with the machinery of canonical loop quantum gravity the we will develop in chapter 4.

2.5 Unitary irreducible representations of $SU(2)$

First of all remember that $SU(2) := \{A \in SL(2,\mathbb{C}) \mid A^\dagger = A^{-1}\}$.

The only thing we must know to apply the Peter-Weyl theory is the classification of the unitary irreps of $SU(2)$.

In this classification plays a fundamental role the space $\mathbb{C}_n[z_1, z_2]$ of the homogeneous polynomials[4] of degree n in the complex variables z_1, z_2.

Every element of this space has the form

$$P(z_1, z_2) = \sum_{k=0}^{n} a_k {z_1}^{n-k} {z_2}^k$$

thus every homogeneous polynomials of degree n can be written as a linear combination of the $n+1$ monomials of degree n

$$Q(z_1, z_2) \equiv z_1^{n-k} {z_2}^k, \qquad k = 0, \ldots, n$$

for this reason $dim(\mathbb{C}_n[z_1, z_2]) = n + 1$.

$\mathbb{C}_n[z_1, z_2]$ is a vector space with the usual operations of sum and multiplication by a complex constant and becomes a Hilbert space when endowed with this inner product:

$$(\sum_{k=0}^{n} a_k {z_1}^{n-k} {z_2}^k \mid \sum_{k=0}^{n} b_k {z_1}^{n-k} {z_2}^k) := \sum_{k=0}^{n} k!(n-k)! a_k \overline{b_k}.$$

The numbers $k!(n-k)!$ are suitable normalization coefficients.

Let now $n \in \mathbb{N} \equiv \{0, 1, 2, \ldots\}$ and consider the following $(n+1)$-dimensional representation of $SU(2)$:

$$\begin{array}{rccl} U^n : & SU(2) & \to & \mathcal{U}(\mathbb{C}_n[z_1, z_2]) \\ & A & \mapsto & {U_A}^n \end{array}$$

[4]Remember that a polynomial is said to be homogeneous if all its terms have the same degree.

where ${U_A}^n$ is defined on the basis of monomials $Q(z_1, z_2)$ in this way

$$({U_A}^n Q)(z_1, z_2) := Q((z_1, z_2).A)$$

and it's extended by linearity to every homogeneous polynomial $P(z_1, z_2)$.

$(z_1, z_2).A$ means the right action[5] of a matrix $A \in SU(2)$ on a row vector $(z_1, z_2) \in \mathbb{C}^2$; if we use the Cayley-Klein parameterization of $A \in SU(2)$, i.e.

$$A = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$$

with $a, b \in \mathbb{C}$, $\mid a \mid^2 + \mid b \mid^2 = 1$, we find that $({U_A}^n Q)(z_1, z_2) = Q((z_1, z_2)A) = Q(az_1 - \bar{b}z_2, bz_1 + \bar{a}z_2)$, and so

$$(U_A^n Q)\,(z_1, z_2) = (az_1 - \bar{b}z_2)^{n-k}(bz_1 + \bar{a}z_2)^k.$$

Theorem 2.5.1 (Classification of the unitary irreps of $SU(2)$) *The representations U^n defined above are unitary and irreducibles. Moreover any other unitary irrep of $SU(2)$ is equivalent to U^n, for a certain $n \in \mathbb{N}$. Thus $\widehat{SU(2)} \simeq \mathbb{N}$.*

From the Peter-Weyl theory it follows immediately that, for every $A \in SU(2)$ and every $f \in L^2(SU(2))$:

$$f(A) = \sum_{n=0}^{\infty} (n+1)\, Tr(\hat{f}(n) {U_A}^n), \qquad \hat{f}(n) = \int_{SU(2)} f(g) {U_{A^{-1}}}^n dA\,.$$

The proof of the last theorem is very technical, but the irreducibility of the representations U^n can be easily worked out by computing the characters χ_{U^n} and verifying (by virtue of theorem 2.3.2) that they have norm 1.

To calculate the $L^2(SU(2))$-norm of the characters χ_{U^n} the first thing to do is to find out the normalized Haar measure on $SU(2)$.

For this scope let's identify $SU(2)$ with the sphere S^3, i.e. the hypersurface of $\mathbb{R}^4$ given by $S^3 := \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid \sum_{i=1}^4 {x_i}^2 = 1\}$; the homeomorphic identification between S^3 and $SU(2)$ is realized by:

$$\begin{array}{rcl} S^3 & \to & SU(2) \\ (x_1, x_2, x_3, x_4) & \mapsto & \begin{pmatrix} x_1 + ix_2 & x_3 + ix_4 \\ -x_3 + ix_4 & x_1 - ix_2 \end{pmatrix} \end{array}$$

[5] Of course we could have used the left action of $SU(2)$ on $\mathbb{C}^2$, putting $({U_A}^n Q)\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} := Q\left(A^{-1}.\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right)$.

Hence, if we identify $\mathbb{R}^4$ with the quaternionic field $\mathbb{H}$ via the isomorphism $(x_1, x_2, x_3, x_4) \to x_1 + ix_2 + jx_3 + kx_4$ then the previous map gives an isomorphism between the multiplicative group $Sp(1)$ of the quaternions of norm 1 and $SU(2)$.

Now let's seek for a convenient parameterization of S^3: since on S^3 we have ${x_1}^2 + {x_2}^2 + {x_3}^2 + {x_4}^2 = 1$, then $-1 \leq x_1 \leq 1$ and so, thanks to the bijection performed by the function *cos* restricted to $[0, \pi]$:

$$\begin{array}{rccc} \cos: & [0,\pi] & \to & [-1,1] \\ & \theta & \mapsto & \cos\theta \end{array}$$

we have that there exists only one $\theta \in [0, \pi]$ such that $x_1 = \cos\theta$, but then $\cos^2\theta + {x_2}^2 + {x_3}^2 + {x_4}^2 = 1$ and so, by the fundamental theorem of trigonometry, ${x_2}^2 + {x_3}^2 + {x_4}^2 = \sin^2\theta$. This shows that the triple (x_2, x_3, x_4) is a point of the 2-dimensional sphere in $\mathbb{R}^3$ of ray $\sin\theta$.

Using the spherical coordinates we can write:

$$\begin{cases} x_2 = \sin\theta\cos\varphi \\ x_3 = \sin\theta\sin\varphi\cos\psi \\ x_4 = \sin\theta\sin\varphi\sin\psi \end{cases}$$

with $\varphi \in [0, \pi]$ and $\psi \in [0, 2\pi]$.

With this parameterization of $S^3 \simeq SU(2)$ the normalized Haar integral of a function $f \in \mathcal{C}(S^3)$ is given by

$$\frac{1}{2\pi^2}\int_0^\pi\int_0^\pi\int_0^{2\pi} f(\theta,\varphi,\psi)\sin^2\theta\sin\varphi\,d\theta\,d\varphi\,d\psi\,.$$

Within this parameterization, the coefficient $\frac{1}{2\pi^2}$ is the correct normalization[6] of the integral, since, taken $f \equiv \mathbf{1}$ one has

$$\int_0^\pi\int_0^\pi\int_0^{2\pi}\sin^2\theta\sin\varphi\,d\theta\,d\varphi\,d\psi = 2\pi\int_0^\pi\sin^2\theta\,d\theta\int_0^\pi\sin\varphi\,d\varphi$$

and an integration by parts gives $2\pi \cdot [\frac{1}{2}(\theta - \sin\theta\cos\theta)]_0^\pi \cdot [-\cos\varphi]_0^{2\pi} = 2\pi \cdot \frac{\pi}{2} \cdot 2 = 2\pi^2$.

[6] If we had used the parameterization: $u \in SU(2)$,

$$u = \cos\frac{\alpha}{2} - i\,\sin\frac{\alpha}{2}\,\vec{n}\cdot\vec{\sigma}$$

where $\vec{n}$ is the unit vector of the rotation axis individuated by u, α is the angle of rotation and $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli matrices, then the normalized Haar measure would be: $\frac{1}{8\pi^2}\sin^2\frac{\alpha}{2}\sin\theta\,d\alpha\,d\theta\,d\varphi$, and so the normalization coefficient would be $\frac{1}{8\pi^2}$.

In [88] (page 53) it's proven that the Haar integral of central functions reduces simply to

$$\frac{2}{\pi}\int_0^\pi f(\theta)\sin^2\theta\, d\theta \qquad f \in \mathcal{C}_{class}(SU(2))\,.$$

At this point the only thing that remains to do is to compute in an explicit way the characters χ_{U^n}. Notice however that every matrix of $SU(2)$ is conjugated to a diagonal matrix and so, being the characters central functions, it's enough to compute them on the diagonal matrices D of $SU(2)$, which are easily recognized to be all of this form:

$$\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$$

for $\theta \in \mathbb{R}$, hence, by direct computation:

$$\chi_{U^n}(D) = \begin{cases} \frac{sin(n+1)\theta}{sin\theta} & \text{if } \theta \neq m\pi \quad m \in \mathbb{Z}; \\ n+1 & \text{if } \theta = m\pi \quad m \in \mathbb{Z}, m \text{ even}; \\ (-1)^n(n+1) & \text{if } \theta = m\pi \quad m \in \mathbb{Z}, m \text{ odd}; \end{cases}$$

being the second option corresponding to the matrix $D = \mathbb{I}_2$ and the third to the matrix $D = -\mathbb{I}_2$.

Since $\mathbb{Z}$ is countable, the contribution of the points $\theta = m\pi$, $m \in \mathbb{Z}$ in the computation of the integral $(\chi_{U^n} \mid \chi_{U^n})$ is zero and so:

$$(\chi_{U^n} \mid \chi_{U^n}) = \frac{2}{\pi}\int_0^\pi \frac{\sin^2(n+1)\theta}{\sin^2\theta}\sin^2\theta\, d\theta = \frac{1}{\pi}\int_0^\pi (1-\cos(2(n+1)\theta))\, d\theta = 1$$

thus $\|\chi_{U^n}\| = 1$ for every $n \in \mathbb{N}$ and the representations U^n are irreducible.

In the physical literature it is custom to label the unitary irreps of $SU(2)$ with the non-negative half integers $j \equiv \frac{n}{2}$, $n \in \mathbb{N}$, instead of the natural numbers n, this is correct since the correspondence $n \leftrightarrow j$ is obviously one-to-one.

It is well known that everyone of the non-negative half integers $j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$, is called the **spin** of the representation. Obviously a spin-j representation has dimension $2j+1$.

We conclude this section by remembering a well known decomposition of the tensor product of two irreps of $SU(2)$.

Theorem 2.5.2 (Clebsch-Gordan decomposition) *Let j_1 and j_2 be the spins of two irreps of $SU(2)$, then*

$$j_1 \otimes j_2 \simeq |j_1 - j_2| \oplus \cdots \oplus (j_1 + j_2).$$

Thus, for example

$$\frac{1}{2} \otimes \frac{1}{2} \simeq 0 \oplus 1$$

as one can verify also with dimensions: the representation spin-$\frac{1}{2}$ has dimension 2, so $\frac{1}{2} \otimes \frac{1}{2}$ has dimension 4, instead the representations spin-0 and spin-1 have dimensions 1 and 3, respectively, thus $0 \oplus 1$ has dimension 4 too.

2.6 Casimir operators

We present here a brief exposition of the Casimir operators and we compute them for the group $SU(2)$. This will be important for the calculation of the spectrum of the volume and area operators in chapter 5.

Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{K}$ and let $T(\mathfrak{g})$ be the tensor algebra of $\mathfrak{g}$, i.e. the algebra whose underlying vector space is

$$T(\mathfrak{g}) = \bigoplus_{n \in \mathbb{N}} \bigotimes^{n} \mathfrak{g}$$

(with $\bigotimes^0 \mathfrak{g} := \mathbb{K}$) and with product defined, for every couple of integers $p, q \geq 0$, by the following formula:

$$(v_1 \otimes \cdots \otimes v_p) \cdot (v_{p+1} \otimes \cdots \otimes v_{p+q}) := v_1 \otimes \cdots \otimes v_{p+q} \qquad \text{if } p, q > 0;$$

$$k \cdot (v_1 \otimes \cdots \otimes v_q) := k(v_1 \otimes \cdots \otimes v_q) \qquad \text{if } p = 0,\ k \in \mathbb{K};$$

and extended on the whole $T(\mathfrak{g})$ by multilinearity.

Let I be the two-sided-ideal of $T(\mathfrak{g})$ generated by the set

$$\{X \otimes Y - Y \otimes X - [X, Y] \mid X, Y \in \mathfrak{g}\}$$

then the quotient algebra

$$\mathfrak{U}(\mathfrak{g}) := T(\mathfrak{g})/I$$

is called the **universal enveloping algebra of $\mathfrak{g}$**.

The universality of $\mathfrak{U}(\mathfrak{g})$ is described in the next theorem.

Theorem 2.6.1 *Let $\mathfrak{g}$ be a Lie algebra, A an associative algebra over $\mathbb{K}$ and let $H \in Hom(\mathfrak{g}, A)$, then there exists $\tilde{H} \in Hom(\mathfrak{U}(\mathfrak{g}), A)$ such that the following diagram commutes*

$$\begin{array}{ccc} \mathfrak{U}(\mathfrak{g}) & \xrightarrow{\tilde{H}} & A \\ {\scriptstyle \pi}\Big\uparrow & & \Big\uparrow{\scriptstyle H} \\ \mathfrak{g} & = & \mathfrak{g} \end{array}$$

where π is the restriction to $\mathfrak{g}$ of the natural projection of $T(\mathfrak{g})$ onto $\mathfrak{U}(\mathfrak{g})$.

Corollary 2.6.1 *If $\mathfrak{g} = Lie(G)$, i.e. $\mathfrak{g}$ is the Lie algebra of a Lie group G, then the following assertions hold:*

1. *π is injective, hence every $X \in \mathfrak{g}$ can be identified with $\pi(X) \in \mathfrak{U}(\mathfrak{g})$;*

2. *$\mathfrak{U}(\mathfrak{g})$ is isomorphic to the Lie algebra of all differential operators on G commuting with the right translations and to the convolution algebra of distributions supported on $\{e_G\}$.*

The universal enveloping algebra is particularly useful when $\mathfrak{g}$ is semisimple, i.e. when the Killing form[7] B is non-degenerate.

In this case, in fact, chosen a basis of $\mathfrak{g}$, $(X_i)_{i=i,\dots,n}$, $n = dim(\mathfrak{g})$, the matrix defined by

$$g_{ij} := B(X_i, X_j)$$

has an inverse $g^{ij} := (g_{ij})^{-1}$ and we can define the **Casimir element** or **Casimir operator** of $\mathfrak{g}$ (when $\mathfrak{U}(\mathfrak{g})$ is identified with the algebra of differential right-invariant operators on G) as

$$\Omega := \sum_{i,j=1}^{n} g^{ij} X_i X_j \quad \in \mathfrak{U}(\mathfrak{g}).$$

The definition is well posed thanks to the next theorem.

Theorem 2.6.2 *Ω is independent from the choice of the basis $(X_i)_{i=1,\dots,n}$ of $\mathfrak{g}$. Moreover $\Omega \in Z(\mathfrak{U}(\mathfrak{g}))$, i.e. Ω commutes with every differential operators on G commuting with the right translations.*

When we specialize $\mathfrak{g}$ to be $\mathfrak{su}(2)$ we have these important result.

Theorem 2.6.3 *The Casimir operator of $\mathfrak{su}(2)$ is*

$$\Omega = \frac{1}{2}(X_1^2 + X_2^2 + X_3^2)$$

and when the identification with the algebra of right-invariant differential operators on G is done, Ω is easily seen to be proportional to the **Laplace operator***:*

$$\Omega = \frac{1}{2}\nabla^2.$$

Moreover the matrix element functions ρ^j_{kl}, where j is the spin of the representation, are eigenfunctions of the differential operator Ω with eigenvalues $j(j+1)$:

$$\Omega(\rho^j_{kl}) = j(j+1)\rho^j_{kl}.$$

[7] The Killing form is the symmetric bilinear form $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{K}$, $B(X,Y) := Tr(ad\,X\,ad\,Y)$, where $ad\,X : \mathfrak{g} \to \mathfrak{g}$, $ad\,X(Y) = [X,Y]$.

2.7 $SL(2,C)$: intrinsic structure and representations

The theory of compact groups, especially $SU(N)$, $N \geq 2$, is of paramount importance for the canonical formulation of loop quantum gravity, but to build up the covariant algorithm one has to handle the more complicated non-compact groups. Fortunately, the most important symmetry group of covariant loop quantum gravity is $SL(2,\mathbb{C})$, the group of the unimodular 2×2 matrices with complex entries, i.e.

$$SL(2,\mathbb{C}) := \{A \in GL(2,\mathbb{C}) \mid det(A) = 1\}$$

whose intrinsic structure and whose representations are well known. This section contains the most important information about this group. The main reference is [58].

$SL(2,\mathbb{C})$ is:

1. *a complex Lie group of dimension 3*;
2. *locally compact*;
3. *unimodular* (the left and the right Haar measures agree);
4. *simply connected*;
5. *reductive*, i.e. $A \in SL(2,\mathbb{C}) \Rightarrow A^\dagger \in SL(2,\mathbb{C})$;
6. *semisimple* (i.e. it's free from non-trivial and non-discrete Abelian subgroups, or, equivalently, its Lie algebra is free from non-trivial Abelian ideals, or, again, its Killing form is non-degenerate);
7. of *finite center*: $Z_{SL(2,\mathbb{C})} = \{\mathbb{I}_2, -\mathbb{I}_2\}$.

Its Lie algebra is the semisimple Lie algebra of the 2×2 traceless matrices with complex entries

$$\mathfrak{sl}(2,\mathbb{C}) := \{X \in \mathfrak{gl}(2,\mathbb{C}) \equiv M(2,\mathbb{C}) \mid Tr(X) = 0\}.$$

A base of $\mathfrak{sl}(2,\mathbb{C})$ is given by the matrices

$$h := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

with brackets: $[h,e] = 2e$, $[h,f] = -2f$, $[e,f] = h$.

Remember now that, if $g_{p,q}$ denotes the metric on $\mathbb{R}^{p+q}$ of signature $p+q$ defined by

$$\begin{array}{rlll} g_{p,q}: & \mathbb{R}^{p+q}\times\mathbb{R}^{p+q} & \to & \mathbb{R} \\ & (x,y) & \mapsto & g_{p,q}(x,y) := x_1y_1+\cdots+x_py_p-x_{p+1}y_{p+1}-\cdots-x_{p+q}y_{p+q}, \end{array}$$

then the **pseudo-orthogonal group** of rank (p,q) is defined to be the subgroup $O(p,q)$ of $GL(p+q,\mathbb{R})$ given by the matrices O that preserves this metric, i.e. such that

$$g_{p,q}(Ox,Oy) = g_{p,q}(x,y) \qquad \forall x,y\in\mathbb{R}^{p+q}.$$

There is an algebraic relation that encodes this feature, namely

$${}^tO\mathbb{I}_{p,q} = \mathbb{I}_{p,q}O^{-1},$$

where

$$\mathbb{I}_{p,q} := \begin{pmatrix} \mathbb{I}_p & 0 \\ 0 & -\mathbb{I}_q \end{pmatrix}.$$

The intersection $O(p,q)\cap SL(p+q,\mathbb{R})$ is indicated by $SO(p,q)$. When $p\equiv 3$ and $q\equiv 1$ one obtains $SO(3,1)$, a non-connected group whose connected component to the identity, $SO_0(3,1)$, is the famous **Lorentz group**.

The relation between $SL(2,\mathbb{C})$ and the Lorentz group is well known: $SL(2,\mathbb{C})$ is the universal covering group of the Lorentz group $SO_0(3,1)$.

The two real forms of $SL(2,\mathbb{C})$ are $SL(2,\mathbb{R})$ and $SU(2)$, this latter is the compact real form and plays a fundamental role in the decompositions of $SL(2,\mathbb{C})$.

We will discuss the Iwasawa decomposition in the last chapter, here we recall only that the **Cartan decomposition** of $SL(2,\mathbb{C})$ is given by the following diffeomorphism:

$$\begin{array}{ccc} SU(2)\times M_H(2) & \longrightarrow & SL(2,\mathbb{C}) \\ (k,H) & \mapsto & k\exp(H) \end{array}$$

where: $M_H(2)$ is the space of the 2×2 hermitian matrices, $A\in GL(2,\mathbb{C})$ such that $A^\dagger = A$, i.e.

$$M_H(2) = \left\{ \begin{pmatrix} x_0+x_3 & x_1+ix_2 \\ x_1-ix_2 & x_0-x_3 \end{pmatrix},\ x_i = 0,\ldots,3\in\mathbb{R} \right\}$$

and, if one considers the metric on $M_H(2)$ induced by the determinant, $det\begin{pmatrix} x_0+x_3 & x_1+ix_2 \\ x_1-ix_2 & x_0-x_3 \end{pmatrix} = {x_0}^2-{x_1}^2-{x_2}^2-{x_3}^2$, one immediately sees that it can be identified with M^4, the (4-dimensional) **Minkowski spacetime**, hence $SL(2,\mathbb{C})$ is, topologically and differentially speaking, the product of the real 3-sphere and the Minkowski spacetime:

$$SL(2,\mathbb{C})\sim SU(2)\times M^4.$$

2.8 Finite-dimensional representations of $SL(2,\mathbb{C})$

Let us begin by listing the finite-dimensional representations of $SL(2,\mathbb{C})$.

An important class of finite-dimensional representation of this group is supported on the space $\mathbb{C}_n[z_1, z_2]$ of the homogeneous polynomials of degree $n \in \mathbb{N}$ in the complex variables z_1 and z_2.

These representations ρ_n are defined as below:

$$\begin{array}{rccc} \rho_n : & SL(2,\mathbb{C}) & \to & Aut(\mathbb{C}_n[z_1, z_2]) \\ & \begin{pmatrix} a & b \\ c & d \end{pmatrix} & \mapsto & \rho_n \begin{pmatrix} a & b \\ c & d \end{pmatrix} \end{array}$$

with

$$\rho_n \begin{pmatrix} a & b \\ c & d \end{pmatrix} P \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} := P \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \right) \equiv P \begin{pmatrix} -bz_2 + dz_1 \\ az_2 - cz_1 \end{pmatrix}.$$

Theorem 2.8.1 *The representations ρ_n described above are irreducible and they are the only holomorphic irreps of $SL(2,\mathbb{C})$ up to equivalence.*

However *the representations ρ_n are not unitary* and one can show that **there aren't finite-dimensional unitary representations of $SL(2,\mathbb{C})$!** This is a big difference with respect to the representation theory of compact groups, where, as we've seen before, the finite-dimensional unitary irreps play a fundamental role in the analysis of the structure of that kind of groups.

Being $SL(2,\mathbb{C})$ simply connected there is a bijection between the set of its finite-dimensional representations and that of its Lie algebra $\mathfrak{sl}(2,\mathbb{C})$: one passes from one to another simply by taking the tangent at the identity.

Moreover remember that $\mathfrak{sl}(2,\mathbb{R})$ and $\mathfrak{su}(2)$ are the real forms of $\mathfrak{sl}(2,\mathbb{C})$, i.e.

$$\mathfrak{sl}(2,\mathbb{C}) = \mathfrak{sl}(2,\mathbb{R}) \oplus i\mathfrak{sl}(2,\mathbb{R}) = \mathfrak{su}(2) \oplus i\mathfrak{su}(2)$$

and the relation between them is stated in the following theorem due to Weyl.

Theorem 2.8.2 (Weyl's unitary trick) *There is a bijection which preserves the invariant subspaces and the equivalence among:*

1. *smooth representations of $SL(2,\mathbb{R})$;*
2. *smooth representations of $SU(2)$;*
3. *holomorphic representations of $SL(2,\mathbb{C})$;*
4. *linear representations of $\mathfrak{sl}(2,\mathbb{R})$;*

5. *linear representations of* $\mathfrak{su}(2)$*;*

6. *complex linear representations, of* $\mathfrak{sl}(2,\mathbb{C})$

where all the representations listed above are implicitly meant to be **finite-dimensional** *and supported on the same complex vector space.*

As a consequence if follows that every holomorphic finite-dimensional representation of $SL(2,\mathbb{C})$ is a direct sum of irreducible representations.

It can be proven that $\mathfrak{sl}(2,\mathbb{C})^{\mathbb{C}} := \mathfrak{sl}(2,\mathbb{C}) \oplus i\mathfrak{sl}(2,\mathbb{C})$, the complexification of $\mathfrak{sl}(2,\mathbb{C})$, is $\mathbb{C}-$isomorphic to $\mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$, which is the complexification of $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$, the Lie algebra of $SU(2) \times SU(2)$.

As a consequence of Weyl's unitary trick and of the representation theory of $SU(2)$, one gets that the finite-dimensional holomorphic representations of $SL(2,\mathbb{C})$ are direct sum of irreducible representations labelled by a couple of natural numbers $(m,n) \in \mathbb{N}^2$ with support space realized by the space of homogeneous polynomials of degree m in the variables (z_1, z_2) and of degree n in the variables $(\bar{z}_1, \bar{z}_2)$ and action given by

$$\rho_{m,n} \begin{pmatrix} a & b \\ c & d \end{pmatrix} P \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} := P\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \right).$$

2.9 Infinite-dimensional representations of $SL(2,\mathbb{C})$

The only unitary non-trivial representations of $SL(2,\mathbb{C})$ are available as infinite-dimensional representations.

Fix a couple of numbers of the type (k, iv), with $k \in \mathbb{Z}$ and $v \in \mathbb{R}$, then the map $\wp^{k,iv} : SL(2,\mathbb{C}) \longrightarrow \mathcal{U}(L^2(\mathbb{C}))$,

$$\wp^{k,iv} \begin{pmatrix} a & b \\ c & d \end{pmatrix} f(z) := |-bz+d|^{-2-iv} \left(\frac{-bz+d}{|-bz+d|} \right)^{-k} f\left(\frac{az-c}{-bz+d} \right)$$

for $f \in L^2(\mathbb{C})$, with respect to the Lebesgue measure on $\mathbb{C}$, is a representation of $SL(2,\mathbb{C})$, moreover the following theorem holds.

Theorem 2.9.1 *For every couple* (k, iv)*,* $k \in \mathbb{Z}$ *and* $v \in \mathbb{R}$*, the corresponding representation* $\wp^{k,iv}$ *is a unitary irrep of* $SL(2,\mathbb{C})$*. Furthermore there is the following unitary equivalence:*

$$\wp^{-k,-iv} \simeq \wp^{k,iv}.$$

It is common to call the family of representations $\wp^{k,iv}$ the **unitary principal series** of $SL(2,\mathbb{C})$.

Besides the unitary principal series there is a family of non-unitary representations of $SL(2,\mathbb{C})$ labelled by the couple (k,w), where:

- $k \in \mathbb{Z}$, $k \neq 0$;
- $w = u + iv \in \mathbb{C}$, $u \neq 0$;

and defined by:

$$\wp^{k,w}\begin{pmatrix} a & b \\ c & d \end{pmatrix} f(z) := |-bz+d|^{-2-w}\left(\frac{-bz+d}{|-bz+d|}\right)^{-k} f\left(\frac{az-c}{-bz+d}\right)$$

where $f \in L^2(\mathbb{C}, (1+|z|^2)^{\Re(w)}dxdy)$.

It can be shown that the $\wp^{k,w}$'s, in the hypothesis above, are not unitary. The family of these representations is called the **non-unitary principal series** of $SL(2,\mathbb{C})$.

However the are particular values of the parameters for which the representations $\wp^{k,w}$ are unitary: in fact when $u = 0$ $\wp^{k,iv}$ reduces to $\wp^{k,iv}$.

When $k = 0$ and $w \in \mathbb{R}$, $0 < w < 2$, then the representation $\wp^{k,w}$ is unitary when one consider the inner product in $L^2(\mathbb{C})$ given by

$$(f|g) := \int_{\mathbb{C}}\int_{\mathbb{C}} \frac{\bar{f}(z)g(\zeta)}{|z-\zeta|^{2-w}} dz d\zeta$$

this family of representation is called the **complementary series**.

The theorem of classification of the irreducible unitary representations of $SL(2,\mathbb{C})$ is the following.

Theorem 2.9.2 *Modulo the equivalences $\wp^{-k,-iv} \simeq \wp^{k,iv}$, the trivial representation, the unitary principal series and the complementary series are the unique irreducible unitary representations of $SL(2,\mathbb{C})$.*

Finally it can be verified that all the irreducible finite-dimensional representations of $SL(2,\mathbb{C})$ are contained in the non-unitary principal series as sub-representations: $\rho_{m,n} \subset \wp^{n-m,-2-m-n}$.

2.10 Plancherel formula for $SL(2,\mathbb{C})$

The Plancherel theorem, already stated for compact groups, is also valid for a wider class of groups, in this section (following [55]) we will state it for a class to which $SL(2,\mathbb{C})$ belongs and (following [58]) we will explicitly show the Plancherel formula of this group.

Consider a locally compact unimodular connected semisimple group G, fix a Haar measure dg on it and a function $f \in L^1(G, dg)$, then, as for compact groups, it's possible to construct the operator-valued function $\hat{f}$ on $\hat{G}$ defined by the formula

$$\hat{f}(\lambda) := \int_G f(g)\rho^\lambda_{g^{-1}} dg$$

where, again, ρ^λ is a generic representative of the class $\lambda \in \hat{G}$. If the support space $\mathcal{H}_\lambda$ of the representation ρ^λ is finite-dimensional, then $\hat{f}(\lambda)$ is simply the analogue of the matrix of the Fourier coefficients which appears in the PW theory of compact groups, but if $\mathcal{H}_\lambda$ is infinite-dimensional, then the operator $\hat{f}(\lambda)$ cannot be represented as a matrix.

However, in our hypothesis for G, every $\mathcal{H}_\lambda$ is separable and the operator $\hat{f}(\lambda)$ is a Hilbert-Schmidt operator on $\mathcal{H}_\lambda$ and so the operator $\hat{f}(\lambda)\hat{f}(\lambda)^\dagger$ is a trace class operator[8] on the same Hilbert space with Hilbert-Schmidt norm given by:

$$\|\hat{f}(\lambda)\|^2_{HS} = Tr(\hat{f}(\lambda)\hat{f}(\lambda)^\dagger).$$

Theorem 2.10.1 (General Plancherel theorem) *Let G be a locally compact unimodular connected semisimple or nilpotent group. Then there exists a measure on $\hat{G}$, called the* **Plancherel measure***, such that*

$$\int_G |f(g)|^2 dg = \int_{\hat{G}} Tr(\hat{f}(\lambda)\hat{f}(\lambda)^\dagger) d\mu(\lambda)$$

or, shortly,

$$\|f\|^2 = \int_{\hat{G}} \|\hat{f}(\lambda)\|^2_{HS}\, d\mu(\lambda)$$

for all $f \in L^1(G, dg) \cap L^2(G, dg)$.

Moreover, defined $L^2(\hat{G}, d\mu)$ to be the Hilbert space of μ-square-integrable operator-valued functions on $\hat{G}$ such that the value at each point $\lambda \in \hat{G}$ is a Hilbert-Schmidt operator on the support space of the representation, the correspondence $f \to \hat{f}$ can be extended to a surjective isometry between $L^2(G, dg)$ and $L^2(\hat{G}, d\mu)$.

[8] In general an operator A on a separable Hilbert space $\mathcal{H}$ is called a **trace class operator** if, for every orthonormal basis $\{h_n\}$ of $\mathcal{H}$, the series $Tr(A) := \sum_n (Ah_n|h_n)$ converges to a finite value, called the **trace** of A, which is independent from the choice of the orthonormal basis.

The Plancherel formula for $SL(2,\mathbb{C})$ is (see [58] for a derivation)

$$\int_{SL(2,\mathbb{C})} |f(g)|^2 dg = c \sum_{k=-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left\| \int_G f(g) \overline{\wp^{k,iv}}(g) dg \right\|_{HS}^2 (k^2 + v^2)\, dv,$$

or, using the general Plancherel theorem,

$$\|f\|^2 = c \sum_{k=-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left\| \hat{f}(k,v) \right\|_{HS}^2 (k^2 + v^2)\, dv$$

for every $f \in L^2(SL(2,\mathbb{C}), dg)$, the constant $c \in \mathbb{R}$ depends on how the Plancherel measure is normalized.

The important fact to stress is that **in the Plancherel formula for $SL(2,\mathbb{C})$ only the unitary principal series appears and *not* the complementary series**.

2.11 Direct integral decompositions

The Plancherel formula for $SL(2,\mathbb{C})$ shows that when we deal with non-compact groups we have to expect not only a discrete part (labelled by the integer parameter k in the previous formula) of the Plancherel measure, but also a continuous part (labelled by the real parameter v). Consequently, the decomposition of the correspondent L^2-space will involve not only an infinite direct sum of Hilbert spaces but, roughly speaking, a continuous analogous of this procedure. This idea is formalized in the concept of **direct integral** and in this section we collect the most important features of this construction.

The theory of direct integrals is originally due to von Neumann [92], in this paper we will follow the beautiful exposition contained in [40].

Let's start defining what we mean by a direct integral of a family of Hilbert spaces $\{\mathcal{H}_\lambda\}_{\lambda \in L}$ w.r.t. a measure μ on the parameter space L.

Roughly speaking, the direct integral of this spaces should be a Hilbert space given by functions f defined on L such that $f(\lambda) \in \mathcal{H}_\lambda$ for each λ and $\int_L \|f(\lambda)\|_\lambda^2 d\mu(\lambda) < \infty$, where $\| \ \|_\lambda$ is the norm of $\mathcal{H}_\lambda$.

To come up with a workable definition of such a space we assume that the parameter space L is a measurable space with measure μ and we introduce some terminology:

- a **sheaf**[9] **of Hilbert spaces** over L is a family of *nonzero separable* Hilbert spaces $\{\mathcal{H}_\lambda\}_{\lambda \in L}$, with inner product $(\ |\ )_\lambda$ and norm $\| \ \|_\lambda$, indexed by L;

[9]Some authors use the term 'field' instead of sheaf.

- a **section** of this sheaf is a map $f : L \to \bigcup_{\lambda \in L} \mathcal{H}_\lambda$ such that $f(\lambda) \in \mathcal{H}_\lambda$ for each $\lambda \in L$;
- a **measurable sheaf of Hilbert spaces** over L is defined to be a couple
$$(\{\mathcal{H}_\lambda\}_{\lambda \in L}, \{e_n\}_{n \in \mathbb{N}})$$
given by a sheaf of Hilbert spaces $\{\mathcal{H}_\lambda\}_{\lambda \in L}$ together with a countable set $\{e_n\}_{n \in \mathbb{N}}$ of sections satisfying the following properties:
    1. the functions $\lambda \mapsto (e_n(\lambda)|e_m(\lambda))_\lambda$ are measurable for all $n, m \in \mathbb{N}$;
    2. the linear span of $\{e_n(\lambda)\}_{n \in \mathbb{N}}$ is dense in $\mathcal{H}_\lambda$ for each λ.

The sections $\{e_n\}_{n \in \mathbb{N}}$ are said to constitute a **fundamental collection** of sections of the sheaf. An elementary example of sheaf that will be important for the later purposes is the **constant sheaf** of a separable Hilbert space $\mathcal{H}$ with orthonormal basis $\{u_n\}$, this is defined by $\mathcal{H}_\lambda = \mathcal{H}$ and $e_n(\lambda) = u_n$ for every λ.

Another easy example is given when L is discrete, i.e. measurable w.r.t. the σ-algebra $\mathcal{P}(L)$, the set of all subsets of L, and μ is the discrete measure; written $d(\lambda) := dim(\mathcal{H}_\lambda)$, given an orthonormal basis $\{e_n(\lambda)\}_{n=1}^{d(\lambda)}$ for each $\mathcal{H}_\lambda$, if we set $e_n(\lambda) = 0$ when $n > d(\lambda)$, then $\{e_n(\lambda)\}_{n \in \mathbb{N}}$ becomes a fundamental collection of sections for the sheaf $\{\mathcal{H}_\lambda\}_{\lambda \in L}$, which, consequently, becomes a measurable sheaf of Hilbert spaces.

Given a measurable sheaf of Hilbert spaces $(\{\mathcal{H}_\lambda\}_{\lambda \in L}, \{e_n\}_{n \in \mathbb{N}})$, we say that **the section f is measurable** if the function

$$\begin{array}{ccc} L & \longrightarrow & \mathbb{C} \\ \lambda & \mapsto & (e_n(\lambda)|f(\lambda))_\lambda \end{array}$$

is measurable for every e_n of the fundamental collection.

Moreover, two sections f and g are said to be **equivalent** is they agree μ-almost everywhere in L.

For the purposes of the direct decompositions, the more interesting sections are the **Hilbert sections**, i.e. those *measurable* sections f such that[10]

$$\int_L \|f(\lambda)\|^2 d\mu < \infty.$$

[10]Note that we write $\|f(\lambda)\|$ and not $\|f(\lambda)\|_\lambda$ because $f(\lambda) \in \mathcal{H}_\lambda$ and so it is obvious (and tacitely assumed) which norm we are considering. The same consideration applies for the inner products.

The space of the *equivalent sections* can be endowed with a natural structure of inner product space with these operations:

$$(f+g)(\lambda) := f(\lambda)+g(\lambda), \qquad \forall\lambda \in L;$$

$$(kf)(\lambda) := kf(\lambda), \qquad k \in \mathbb{C}, \forall\lambda \in L;$$

$$(f|g) := \int_L (f(\lambda)|g(\lambda))d\mu.$$

It can be proved (see, e.g., [33]) that this space is *complete* w.r.t. the norm generated by the inner product, hence it is an Hilbert space; in particular it is the Hilbert space we were looking for, namely it's the **direct integral** of the sheaf $(\{\mathcal{H}_\lambda\}_{\lambda\in L}, \{e_n\}_{n\in\mathbb{N}})$ w.r.t. the measure μ and its symbol is:

$$\int_L^{\oplus} \mathcal{H}_\lambda \, d\mu.$$

The generic element of $\int_L^{\oplus} \mathcal{H}_\lambda \, d\mu$ defined by the section $\lambda \mapsto f(\lambda)$ will be denoted with $\int_L^{\oplus} f(\lambda) d\mu$.

It can also be proved (see again [33]) that, if L is a locally compact 2nd-countable Hausdorff space and μ is a positive Borel measure, then the direct integral $\int_L^{\oplus} \mathcal{H}_\lambda \, d\mu$ is a separable Hilbert space.

In particular, in the case of a constant sheaf $\mathcal{H}_\lambda \equiv \mathcal{H}$ for every $\lambda \in L$, the Hilbert sections identify with the measurable functions $f : L \to \mathcal{H}$ such that $\int_L |f|^2 d\mu < \infty$, the Hilbert space they compose is denoted by $L^2(L, \mu; \mathcal{H})$, hence:

$$\int_L^{\oplus} \mathcal{H}_\lambda \, d\mu = L^2(L, \mu; \mathcal{H}).$$

If a direct integral $\int_L^{\oplus} \mathcal{H}_\lambda \, d\mu$ of a sheaf of Hilbert spaces $\{\mathcal{H}_\lambda\}_{\lambda\in L}$ is separable, then we define an **orthonormal basis of sections of** $\int_L^{\oplus} \mathcal{H}_\lambda \, d\mu$ to be a countable[11] collection of measurable sections $\{u_n\}_{n\in\mathbb{N}}$ such that:

- if $dim(\mathcal{H}_\lambda) = \aleph_0$, then $\{u_n(\lambda)\}_{n\in\mathbb{N}}$ is an orthonormal basis of $\mathcal{H}_\lambda$ for every $\lambda \in L$;

- if $dim(\mathcal{H}_\lambda) < \aleph_0$, then $\{u_n(\lambda)\}_{n=1,2,\ldots,dim(\mathcal{H}(\lambda))}$ is an orthonormal basis for $\mathcal{H}_\lambda$ and $u_n(\lambda) = 0$ for $n > dim(\mathcal{H}_\lambda)$.

It can be proved that orthonormal bases of sections of a direct integral of Hilbert spaces always exist and that they have the same role as the usual

[11] Remember that in a separable Hilbert space all the orthonormal basis are countable.

orthonormal bases of Hilbert spaces. In particular, for every $f, g \in \int_L^{\oplus} \mathcal{H}_\lambda \, d\mu$ the following expansions hold:

$$f = \int_L^{\oplus} \sum_{n\in\mathbb{N}} (u_n(\lambda)|f(\lambda))\, u_n(\lambda)\, d\mu \;;$$

$$(f|g) = \int_L^{\oplus} \sum_{n\in\mathbb{N}} (f(\lambda)|u_n(\lambda))\, (u_n(\lambda)|g(\lambda))\, d\mu \;.$$

2.11.1 Direct integral of operators on Hilbert spaces

Let $(\{\mathcal{H}_\lambda\}_{\lambda\in L}, \{e_n\}_{n\in\mathbb{N}})$ be a measurable sheaf of Hilbert spaces on L, then a **sheaf of operators** $O(\lambda) \in \mathcal{B}(\mathcal{H}_\lambda)$, for every $\lambda \in L$, will be called **measurable** if, for every measurable section f of the sheaf $\mathcal{H}_\lambda$, the section

$$\begin{array}{ccc} L & \longrightarrow & \mathcal{H}_\lambda \\ \lambda & \mapsto & O(\lambda)f(\lambda) \end{array}$$

is measurable.

Moreover a sheaf of operators $O(\lambda) \in \mathcal{B}(\mathcal{H}_\lambda)$, for every $\lambda \in L$, will be called **essentially bounded** if the measurable function

$$\begin{array}{ccc} L & \longrightarrow & \mathbb{R} \\ \lambda & \mapsto & \|O(\lambda)\| \end{array}$$

is essentially bounded[12].

If $L \ni \lambda \mapsto O(\lambda) \in \mathcal{B}(\mathcal{H}_\lambda)$ is a measurable and essentially bounded sheaf of operators and f is a Hilbert section, then the section g defined by $g(\lambda) := O(\lambda)f(\lambda)$ is also a Hilbert section and so the measurable and essentially bounded sheaf of operators

$$\begin{array}{ccc} L & \longrightarrow & \mathcal{B}(\mathcal{H}_\lambda) \\ \lambda & \mapsto & O(\lambda) \end{array}$$

induces an bounded linear operator O on the direct integral $\int_L^{\oplus} \mathcal{H}_\lambda \, d\mu$, which we denote with

$$\int_L^{\oplus} O(\lambda) d\mu$$

[12] Given a measurable space X, a measurable function $\varphi : X \to \mathbb{R}$ is called essentially bounded if there exists a real constant $M > 0$ such that $|\varphi(x)| \leq M$ almost everywhere in X.

and which we define in a obvious way as:

$$\left[\left(\int_L^{\oplus} O(\lambda)d\mu\right) f\right](\lambda) := O(\lambda)f(\lambda) \qquad \forall\lambda \in L.$$

Elementary algebraic properties of the direct integral of operators are:

$$\int_L^{\oplus} O(\lambda)d\mu + \int_L^{\oplus} O'(\lambda)d\mu = \int_L^{\oplus} (O+O')(\lambda)d\mu \ ;$$

$$\left(\int_L^{\oplus} O(\lambda)d\mu\right)\left(\int_L^{\oplus} O'(\lambda)d\mu\right) = \int_L^{\oplus} (OO')(\lambda)d\mu \ ;$$

$$\left(\int_L^{\oplus} O(\lambda)d\mu\right)^{\dagger} = \int_L^{\oplus} O^{\dagger}(\lambda)d\mu \ .$$

Moreover it can be proved that

$$\left\|\int_L^{\oplus} O(\lambda)d\mu\right\| = \text{ess sup}_{\lambda\in L}\|O(\lambda)\| < \infty.$$

A simple but important case of direct integral of operators arises when the operators $O(\lambda)$ of the sheaf are all scalar multiples of the identity, i.e. when there is a function[13] $\varphi \in L^{\infty}(L,\mu)$ such that the the sheaf satisfies $O_{\varphi}(\lambda) \equiv \varphi(\lambda)\mathbf{1}(\lambda)$. Such operators on $\int_L^{\oplus} \mathcal{H}_{\lambda}\, d\mu$ are called **diagonal operators**. In particular, if the function φ belongs to $\mathcal{C}_0(L)$, i.e. is a continuous function which vanishes at infinity, then the previous operator is called **continuously diagonal**.

Notice that the diagonal operators on $\int_L^{\oplus} \mathcal{H}_{\lambda}\, d\mu$ are bounded multiplications operators on this space, these constitute the generalization of the bounded multiplications operators on the Hilbert spaces of kind $L^2(\mathcal{M},\mu)$, where $\mathcal{M}$ is a measurable space.

The generalization of the unbounded multiplication operators on such a spaces is realized following the same construction as above and dropping out the condition of essential boundness on φ, i.e. by requiring only its measurability.

The set of the diagonal operators and the continuously diagonal operators, under the obvious algebraic operations, is an Abelian $*$-subalgebra[14] of

[13] $L^{\infty}(L,\mu)$ is the Banach space of the essentially bounded measurable functions on X with norm given by

$$\|\varphi\|_{\infty} := \text{ess sup}(\varphi) \equiv \text{Inf}\{M > 0 \text{ such that } \varphi \text{ is essentially bounded on } X\}.$$

[14] For the theory of $*$-algebras, C^*-algebras and von Neumann algebras see the next chapter.

$\mathcal{B}(\int_L^\oplus \mathcal{H}_\lambda \, d\mu)$ and it can be proved that they are $*$-isomorphic to $L^\infty(L,\mu)$ and $\mathcal{C}_0(L)$, respectively.

Both these $*$-algebras are closed w.r.t. the operator-norm topology and so they are Abelian C^*-subalgebras of $\mathcal{B}(\int_L^\oplus \mathcal{H}_\lambda \, d\mu)$. Furthermore it can be proved that the C^*-algebra of the diagonal operators on $\int_L^\oplus \mathcal{H}_\lambda \, d\mu$ is always unital weakly closed, i.e. it's closed w.r.t. the weak topology of $\mathcal{B}(\int_L^\oplus \mathcal{H}_\lambda \, d\mu)$.

It follows that **the diagonal operators on $\int_L^\oplus \mathcal{H}_\lambda \, d\mu$ form an Abelian von Neumann algebra**.

The direct integral of operators finds important applications in the theory of the diagonalization of normal operators on a Hilbert space.

2.11.2 Direct integral of representations and Plancherel decomposition

Fix G to be a given locally compact group and denote with ρ^λ a generic unitary representation of G on the Hilbert space $\mathcal{H}_\lambda$.

If, for each $\lambda \in L$ and each $g \in G$, the map $L \ni \lambda \mapsto \rho^\lambda(g) \in \mathcal{B}(\mathcal{H}_\lambda)$ is a measurable sheaf of operators, then the family $\{\rho^\lambda(g)\}_{\lambda \in L}$ will be called a **measurable sheaf of representations** of G.

If this is the case then, since the unitary operators $\rho^\lambda(g)$ have norm 1, the sheaf is uniformly bounded and so we can form the direct integral

$$\rho(g) := \int_L^\oplus \rho^\lambda(g) d\mu$$

which is obviously called the **direct integral of the representations** ρ^λ and it can be easily proved to be a unitary representation of G on the Hilbert space $\int_L^\oplus \mathcal{H}_\lambda \, d\mu$:

$$\begin{array}{rccl} \rho: & G & \longrightarrow & \mathcal{U}(\int_L^\oplus \mathcal{H}_\lambda d\mu) \\ & g & \mapsto & \rho(g) := \int_L^\oplus \rho^\lambda(g) \, d\mu. \end{array}$$

The following theorem asserts, among other things, that every unitary representation of a wide class of locally compact groups is equivalent to a direct integral of representations of the same group.

Theorem 2.11.1 *Let G be a 2^{nd}-countable locally compact group, let ρ be a unitary representation of G on a separable Hilbert space $\mathcal{H}$ and let $\mathfrak{A}$ be a weakly closed Abelian C^*-subalgebra of Int(ρ)*[15]*. Then there is a standard*

[15] With Int(ρ) we denote the von Neumann algebra generated by the intertwiners between the representation ρ on $\mathcal{H}$ and itself.

measure space $(L, \mathcal{M}, \mu)$, a measurable sheaf of Hilbert spaces $\{\mathcal{H}_\lambda\}$ and a unitary map $\mathfrak{U} : \mathcal{H} \to \int_L^\oplus \mathcal{H}_\lambda$ such that:

1. $\mathfrak{U}\,\rho^\lambda\,\mathfrak{U}^{-1} = \int_L^\oplus \rho^\lambda(g) d\mu \qquad \forall g \in G$;
2. $\mathfrak{U}\mathfrak{A}\mathfrak{U}^{-1}$ *is the algebra of the diagonal operators on* $\int_L^\oplus \mathcal{H}_\lambda \, d\mu$.

The rest of this section is dedicated to the explicit decomposition of $L^2(G)$, where G is a fixed *locally compact,* 2^{nd}*-countable, unimodular, semisimple, connected group* G as a direct integral of irreducible representations.

In this decomposition will play a fundamental role the right and left regular representations of G on $L^2(G)$ (w.r.t. a given Haar measure), remember that these representations are defined by: $(R_g\, f)(h) := f(hg)$, and $(L_g\, f)(h) := f(g^{-1}h), \forall f \in L^2(G)$. Remember also that they can be combined to give the two-sided regular representation, a unitary representation of $G \times G$ on $L^2(G)$ defined by: $(\tau(g,h)\, f)(k) := f(h^{-1}kg), \forall f \in L^2(G)$.

We are now ready to apply the results of the direct integral decompositions to obtain a Plancherel decomposition. First of all we identify the measurable space L over which we take the direct integral with the dual object $\hat{G}$ of G, we denote again its elements (equivalence classes of unitary irreps of G) with λ and we write ρ^λ for a generic representative of the class λ.

Since the unitary irreps of the locally compact groups need not to be finite-dimensional, as happens for compact groups, we must remember that, if the support space $\mathcal{H}_\lambda$ of a representation is infinite-dimensional, then the tensor product space $\mathcal{H}_\lambda \otimes \overline{\mathcal{H}_\lambda}$ is a proper subspace[16] of $\mathcal{B}(\mathcal{H}_\lambda)$ which consists of the Hilbert-Schmidt operators on $\mathcal{H}_\lambda$, i.e.

$$\mathcal{H}_\lambda \otimes \overline{\mathcal{H}_\lambda} \simeq \mathcal{B}_{HS}(\mathcal{H}_\lambda).$$

Define now, for a function $f \in L^1(G)$ its **Fourier transform** as a measurable sheaf of operators on $\hat{G}$ given by these operator-valued integrals:

$$\hat{f}(\lambda) := \int_G f(g)\rho^\lambda(g^{-1})dg$$

which can be proved to belong to $\mathcal{B}_{HS}(\mathcal{H}_\lambda)$ for every λ.

With these notations we have the following important theorem essentially due to Segal [87].

Theorem 2.11.2 (Plancherel direct decomposition) *Fixed a Haar measure on G, there exists a unique measure μ, called the* **Plancherel measure***, on $\hat{G}$ with the following properties:*

[16] In the finite dimensional case the two spaces agree.

1. *the Fourier transform extends to a unitary map*

$$\begin{array}{cccc} \mathcal{F}: & L^2(G) & \longrightarrow & \int_{\hat{G}}^{\oplus} \mathcal{H}_\lambda \otimes \overline{\mathcal{H}_\lambda}\, d\mu \\ & f & \mapsto & \hat{f} \end{array}$$

2. *the Fourier transform is an intertwining operator for the two-sided regular representation τ and the direct integral representation $\int_{\hat{G}}^{\oplus} \rho^\lambda \otimes \overline{\rho^\lambda} d\mu$.*

We can summarize the theorem with these two unitary equivalences:

$$L^2(G) \simeq \int_{\hat{G}}^{\oplus} \mathcal{H}_\lambda \otimes \overline{\mathcal{H}_\lambda}\, d\mu \ ;$$

$$\tau \simeq \int_{\hat{G}}^{\oplus} \rho^\lambda \otimes \overline{\rho^\lambda} d\mu \ .$$

Segal's theorem is immediately seen to be a generalization of the classical Plancherel theorem and the Peter-Weyl theorem for locally compact Abelian groups and compact groups, respectively. For locally compact Abelian groups the classical Plancherel theorem implies that the Plancherel measure is nothing but the Haar measure (conveniently normalized) on the dual group. For compact groups the Peter-Weyl theorem implies that the Plancherel measure relative to the normalized Haar measure on G is the measure that assigns to each class $\lambda \in \hat{G}$ the mass $dim(\lambda)$.

Apart from the presence of the direct integral, the very big difference between the locally compact Abelian groups (or the compact groups) and the locally compact ones is the fact that the support of the Plancherel measure, for these kind of groups, *can be a proper subset of* $\hat{G}$, indicated by $\hat{G}_r$ and called the **reduced dual** of G.

There is a class of locally compact groups for which the reduced dual $\hat{G}_r$ agrees with the whole $\hat{G}$, they are called **amenable groups**. It's known that the non-compact semisimple groups are not amenable, so, for example, $SL(2,\mathbb{C})$ is not amenable and in fact its Plancherel measure has support only on the unitary principal series and not on the complementary series, which constitute a subset of zero Plancherel measure in $\widehat{SL(2,\mathbb{C})}$.

Harish-Chandra was able to determine the Plancherel measure on every connected semisimple Lie groups even though in many cases the full dual object of these groups is not precisely characterized, but, of course, the 'unknown' representations form a set of Plancherel measure zero.

Some further results related to the Plancherel theorem are afforded by the following theorem due again to Segal [86].

Theorem 2.11.3 *Let G be a unimodular locally compact group with regular representations R and L and let $\mathcal{R}$ and $\mathcal{L}$ be the von Neumann algebras in $\mathcal{B}(L^2(G))$ generated by $\{R_g \mid g \in G\}$ and $\{L_g \mid g \in G\}$, respectively. Then:*

1. *$\mathcal{R} = Int(L)$ and $\mathcal{L} = Int(R)$;*
2. *an operator $T \in \mathcal{B}(L^2(G))$ commutes with every element of $Int(R)$ iff $T \in Int(L)$, and viceversa;*
3. *$Int(\tau)$ is precisely the common center of $Int(R)$ and $Int(L)$.*

Chapter 3

Theory of Abelian C*-algebras

3.1 Introduction: basic definitions

Together with the theory of fiber bundles, group representations and direct decompositions, the theory of Abelian C^*-algebras is of vital importance in many steps of the algorithm of canonical loop quantization of gravity, in this chapter we present the most important information about C^*-algebras that we will use in the following chapter to build up the framework of canonical loop quantum gravity.

Let us begin by remembering some terminology and basic facts about algebras.

An algebra $\mathfrak{A}$ over the field $\mathbb{K}$ is a vector space $\mathfrak{A}$ over $\mathbb{K}$ endowed with an internal binary operation, called **product** and indicated simply by juxtaposing the elements of $\mathfrak{A}$, which is compatible with the linear structure of $\mathfrak{A}$, i.e. this operation is bilinear:

1. $a(b + c) = ab + ac$;
2. $(a + b)c = ac + bc$;
3. $k(ab) = (ka)b = a(kb)$;

$\forall a, b, c \in \mathfrak{A}$ and $\forall k \in \mathbb{K}$.

CONVENTION: In what follows we will be interested only in the analysis of **complex algebras**, i.e. those for which $\mathbb{K} \equiv \mathbb{C}$, thus 'algebra' will always be intended to be over the complex field.

A first qualification of algebras can be given by analyzing the properties of the product, as listed below.

Let $\mathfrak{A}$ be an algebra.

- If the product of $\mathfrak{A}$ is associative, i.e.:

$$a(bc) = (ab)c \qquad \forall a, b, c \in \mathfrak{A},$$

then $\mathfrak{A}$ is said to be an **associative algebra**. All the algebras considered in the following will be tacitly assumed associative;

- If there exists an element $u \in \mathfrak{A}$ such that $ua = au = a$, $\forall a \in \mathfrak{A}$, then u is said to be the unity of $\mathfrak{A}$, which is called an **unital algebra** (or algebra with unit u). Another symbol widely used to denote the unity of an algebra is e, because it reminds the German word 'Einselement', which means 'unit element';

- If $ab = ba$, $\forall a, b \in \mathfrak{A}$, then $\mathfrak{A}$ is an **Abelian** or **commutative algebra**;

- a **homomorphism of algebras**, $\mathfrak{A}$ and $\mathfrak{B}$, is a linear map $\varphi : \mathfrak{A} \to \mathfrak{B}$ which is multiplicative:

$$\varphi(ab) = \varphi(a)\varphi(b);$$

- let's also remember the substructures of an algebra: given an algebra $\mathfrak{A}$, a **subalgebra** of $\mathfrak{A}$ is a vector subspace $\mathfrak{A}'$ of $\mathfrak{A}$ stable w.r.t. the restriction of the product of $\mathfrak{A}$ to the elements of $\mathfrak{A}'$. A left (resp. right) **ideal** $\mathfrak{I}$ of $\mathfrak{A}$ is a subalgebra of $\mathfrak{A}$ with the stronger property that:

$$\text{if } a \in \mathfrak{A} \text{ and } \ b \in \mathfrak{I} \ \Rightarrow \ ab \in \mathfrak{I} \text{ (resp. } ba \in \mathfrak{I}).$$

If $\mathfrak{I}$ is, at the same time, a left and right ideal of $\mathfrak{A}$, then it is called a bilateral ideal, or simply ideal, of $\mathfrak{A}$;

- given the algebra $\mathfrak{A}$ and an ideal $\mathfrak{I}$, it's possible to construct the quotient algebra $\mathfrak{A}/\mathfrak{I}$ in this way:

  - $\mathfrak{A}/\mathfrak{I}$, as a set, is the set of equivalence classes of elements of $\mathfrak{A}$ w.r.t. the equivalence relation:

  $$a \sim b \ \Leftrightarrow \ \exists c \in \mathfrak{I} \text{ such that: } a = b + c.$$

  We denote the class to which a belongs with $(a + \mathfrak{I})$;

  - the linear structure of $\mathfrak{A}/\mathfrak{I}$ is:

  $$\begin{cases} (a+\mathfrak{I}) + (b+\mathfrak{I}) := ((a+b)+\mathfrak{I}); \\ k(a+\mathfrak{I}) := (ka+\mathfrak{I}); \end{cases}$$

- finally the algebraic structure of $\mathfrak{A}/\mathfrak{I}$ is given by the product:

$$(a+\mathfrak{I})(b+\mathfrak{I}) := (ab+\mathfrak{I})\,;$$

- If $\mathfrak{A}$, as a vector space, is normed with a norm $\|\ \|$ satisfying the 'submultiplicative condition':

$$\|ab\| \leq \|a\| \cdot \|b\| \qquad \forall a, b \in \mathfrak{A}$$

  then $\mathfrak{A}$ is said to be a **normed algebra**. The submultiplicative condition assures that the product $(a,b) \to ab$ in jointly continuous w.r.t. the topology generated by the norm, in fact, thanks to the triangular inequality we have

$$\|ab - a'b'\| = \|a(b-b') - (a-a')b'\| \leq \|a\|\|b-b'\| + \|a-a'\|\|b'\| \to 0$$

  if $a \to a'$, $b \to b'$;

- if $\mathfrak{A}$ is a normed algebra with unit, then it's always possible to find a norm equivalent to the starting one (i.e. inducing the same topology) such that $\|u\| = 1$. For this reason it's custom to assume this condition as implicit in the definition of a normed algebra with unit;

- $\mathfrak{A}$ is a **Banach algebra** if it is a normed algebra and if, as normed vector space, is complete in the topology generated by its norm (i.e. every Cauchy sequence in $\mathfrak{A}$ is also convergent in $\mathfrak{A}$ w.r.t. that topology).

EXAMPLES.

The following are the most important examples of Banach algebras of functions. In every example, unless otherwise stated, the algebraic operations are pointwise defined.

1. If X is any topological space, then $\mathcal{C}_b(X)$, the space of complex-valued bounded continuous functions on it, is an Abelian Banach algebra w.r.t. the norm $\|f\|_\infty := \sup_{x\in X}|f(x)|$;

2. $\mathcal{C}_0(X)$, the space of the complex-valued continuous functions on a locally compact Hausdorff topological space X that vanish at infinity is a very important example of Abelian Banach algebra (w.r.t. $\|\ \|_\infty$), their characteristic property is the following:

$$\forall \epsilon > 0 \ \exists K_\epsilon \text{ (compact in } X\text{) such that: } \ |\, f(x)\,| < \epsilon \ \ \forall x \in {K_\epsilon}^c\,.$$

   $\mathcal{C}_0(X)$ is a subalgebra of $\mathcal{C}_b(X)$ and it has unit if and only if X is compact, in that case $\mathcal{C}(X) = \mathcal{C}_b(X) = \mathcal{C}_0(X)$ and the unit is the constant function equal to 1;

3. If (M, μ) is any measure space, then $L^\infty(M, \mu)$, the space of complex-valued essentially bounded on M, i.e. satisfying $|f(x)| \leq c$, for a certain $c \in \mathbb{R}^+$, μ-almost everywhere in M, is an Abelian Banach unital algebra w.r.t. the norm $\|f\|_\infty := Inf\{c\}$, where c is any number satisfying the previous inequality;

4. If G is any locally compact topological group, then it admits a Haar measure μ_H and we can consider the Banach space $L^1(G, \mu_H)$, this is an Abelian Banach algebra w.r.t. the convolution

$$(f * g)(x) := \int_G f(y^{-1}x)g(y)d\mu_H(y).$$

It has unit if and only if G is discrete, in that case the unit is the characteristic function of the identity of G.

If an algebra is not unital, one can always embed it into a unital algebra by following this standard procedure:

UNITALIZATION OF AN ALGEBRA.

Let $\mathfrak{A}$ be an algebra without unit, we want to build an algebra $\tilde{\mathfrak{A}}$ with unit that contains $\mathfrak{A}$ as subalgebra:

- as vector space it is $\tilde{\mathfrak{A}} := \mathfrak{A} \oplus \mathbb{C}$;
- its multiplication is

$$\begin{array}{ccc} \tilde{\mathfrak{A}} \times \tilde{\mathfrak{A}} & \longrightarrow & \tilde{\mathfrak{A}} \\ ((a,\lambda),(b,\eta)) & \mapsto & (ab + \lambda b + \eta a, \lambda\eta) \end{array}$$

- the unit element of $\tilde{\mathfrak{A}}$ is $(e, 1)$, where e is the unit of $\mathfrak{A}$ viewed as vector space;
- the embedding of $\mathfrak{A}$ in $\tilde{\mathfrak{A}}$ is given by the homomorphism

$$\begin{array}{ccc} \mathfrak{A} & \longrightarrow & \tilde{\mathfrak{A}} \\ a & \hookrightarrow & (a, 0); \end{array}$$

- if $\mathfrak{A}$ is a Banach algebra w.r.t. the norm $\| \ \|$ then $\tilde{\mathfrak{A}}$ is also a Banach algebra w.r.t. the norm $(a, \lambda) := \|a\| + |\lambda|$.

For example, one can verify that the unitalization of $\mathcal{C}_0(X)$, where X is a locally compact Hausdorff space, gives $\mathcal{C}_\infty(X)$, i.e. the Banach algebra of continuous functions on X that admit limit at infinity.

The next key step toward the analysis of C^*-algebras is to mix consistently the normed structure of a Banach algebra with a $*$-structure. Precisely, let $\mathfrak{A}$ be an algebra, then:

- $\mathfrak{A}$ is said to be a $*$-algebra if there exists a map

$$^* : \mathfrak{A} \to \mathfrak{A}$$

  satisfying the following conditions:

  1. $(a+b)^* = a^* + b^*$;
  2. $(ka)^* = \bar{k}a^*$;
  3. $(ab)^* = b^*a^*$;
  4. $a^{**} = a$;

  $\forall a, b \in \mathfrak{A}$ and $\forall k \in \mathbb{C}$. The application * is called **involution**;

- $a \in \mathfrak{A}$ is **self-adjoint** if: $a^* = a$;

- the most connatural maps with the $*$-algebraic structure are the so-called **$*$-homomorphisms**: given two $*$-algebras $\mathfrak{A}$ and $\mathfrak{B}$, then an algebraic homomorphism $\varphi : \mathfrak{A} \to \mathfrak{B}$ is called $*$-homomorphism if:

$$\varphi(a^*) = [\varphi(a)]^*,$$

  where the $*$ at the left hand side is the involution of $\mathfrak{A}$ and the one that appears at the right hand side is the involution of $\mathfrak{B}$.

- if an $*$-algebra $\mathfrak{A}$ is normed with norm $\| \ \|$ and the condition:

$$\|a^*\| = \|a\| \qquad \forall a \in \mathfrak{A}$$

  holds, then $\mathfrak{A}$ is called a **normed $*$-algebra**;

- if $\mathfrak{A}$ is a normed $*$-algebra and, at the same time, a Banach algebra, then $\mathfrak{A}$ is called a **Banach $*$-algebra**.

There is an apparently 'innocent' condition that relates the norm and the $*$-operation in a Banach $*$-algebra that has powerful consequences on its structure, this is the condition that defines a C^*-algebra: a Banach $*$-algebra $\mathfrak{A}$ satisfying the condition:

$$\|a^*a\| = \|a\|^2 \qquad \forall a \in \mathfrak{A}$$

is called a **C^*- algebra**.

Let's see some examples of C^*-algebras.

- $\mathbb{C}$ is an Abelian C^*-algebra with unit, involution given by the complex conjugation and norm given by the usual Euclidean norm on $\mathbb{C}$;

- $\mathcal{B}(\mathcal{H})$, the algebra of all bounded (alias continuous) linear operator on a Hilbert space $\mathcal{H}$, is a C^*-algebra with unit (the identity operator), in general non-Abelian, with involution given by the adjoint operation and operator norm:

$$\|T\| := \sup_{x\neq 0}\{\frac{\|Tx\|}{\|x\|}\} \qquad T \in \mathcal{B}(\mathcal{H});$$

- $\mathcal{C}(X)$, X compact Hausdorff space, is an Abelian C^*-algebra with unit, the involution is given by the complex conjugation pointwise defined for every map $f \in \mathcal{C}(X)$ and the norm is $\|f\|_\infty$;

- $\mathcal{C}_b(X)$, X locally compact Hausdorff space, is an Abelian C^*-algebra without unit. The involution and the norm are the same as those of $\mathcal{C}(X)$.

3.2 General results on Abelian Banach algebras

In the analysis of the structure of the Banach algebras with unit u it has a great importance the concept of **spectrum of an element** a of $\mathfrak{A}$; this is the subset of $\mathbb{C}$ defined by

$$\sigma(a) := \{\lambda \in \mathbb{C} \mid \lambda u - a \text{ is not invertible}\}.$$

This is a well known concept if we think at the algebra of the $N \times N$ matrices with complex entries, where the spectrum of a matrix is the set of its eigenvalues.

The complementary set of the spectrum of a is called the **resolvent** of a: $\mathrm{Res}(a) := \mathbb{C} \setminus \sigma(a)$.

With techniques analogues to the ones used in the particular case of $\mathcal{B}(\mathcal{H})$, one can prove the following theorem.

Theorem 3.2.1 *If $\mathfrak{A}$ is a Banach algebra with unit and $a \in \mathfrak{A}$, then:*

1. *$\sigma(a)$ is a closed subset of the closed disk of ray $\|a\|$ in $\mathbb{C}$, hence $Res(a)$ is an open set in $\mathbb{C}$;*

2. *the map*

$$\begin{array}{ccc} Res(a) & \longrightarrow & \mathfrak{A} \\ \lambda & \mapsto & (\lambda u - a)^{-1} \end{array}$$

is holomorphic.

We can now state the most significative results about Banach algebras with unit, which shows that $\mathbb{C}$ is the prototype of an entire class of such algebras.

Theorem 3.2.2 (Gelfand-Mazur) *Let $\mathfrak{A}$ be a Banach algebra with unit u, then:*

1. $\sigma(a) \neq \emptyset$;
2. *if every non zero element of $\mathfrak{A}$ has an inverse, then $\mathfrak{A}$ is isomorphic to $\mathbb{C}$.*

We can now reach more results about the structure of Banach algebras if we define the concept of the spectrum of the *whole* algebra.

Let then $\mathfrak{A}$ be a Banach algebra. First of all define a character of $\mathfrak{A}$ as a *non-identically zero* linear functional ϕ on $\mathfrak{A}$ such that:

$$\phi(ab) = \phi(a)\phi(b) \qquad \forall a, b \in \mathfrak{A}.$$

The **spectrum of an Banach algebra** is the set of all its characters. The symbol used to denote it is $\sigma(\mathfrak{A})$.

The next theorem contains the most important features about the characters of an Abelian Banach algebra with unit. Let's begin with a useful technical lemma.

Lemma 3.2.1 *Let $\mathfrak{A}$ be an Abelian Banach algebra with unit u and let I be a maximal ideal of $\mathfrak{A}$, i.e. a bilateral ideal not contained in any other ideal of $\mathfrak{A}$. Then:*

- *I is closed;*
- *the quotient $\mathfrak{A}/I$ is an Abelian unital Banach algebra with norm*

$$\|(a+I)\| := \inf_{b \in I} \|a+b\|.$$

Moreover it satisfies the hypothesis of the Gelfand-Mazur theorem, thence:

$$\mathfrak{A}/I = \mathbb{C}(u+I).$$

Theorem 3.2.3 *Let $\mathfrak{A}$ be an Abelian Banach algebra with unit u and let ϕ be a character of $\mathfrak{A}$. Then:*

1. *ϕ is bounded and has unit norm: $\|\phi\| = 1$;*

2. *$\sigma(\mathfrak{A})$ is a compact Hausdorff space in the w^*-topology*[1] *relativized to it, usually called* **Gelfand topology***;*

3. *denoted with $Max(\mathfrak{A})$ the set of the maximal ideals of $\mathfrak{A}$, the map*

$$\begin{array}{ccc} \sigma(\mathfrak{A}) & \longrightarrow & Max(\mathfrak{A}) \\ \phi & \mapsto & Ker(\phi) \end{array}$$

   is a bijection;

4. *$\forall a \in \mathfrak{A}$, $\sigma(a) = \{\phi(a) \mid \phi \in \sigma(\mathfrak{A})\}$, hence if we know the spectrum of $\mathfrak{A}$ we can calculate the spectrum of each element of $\mathfrak{A}$...this is obviously the reason for the name 'spectrum of $\mathfrak{A}$'.*

If $\mathfrak{A}$ doesn't have unit, then it can be proved that its spectrum $\sigma(\mathfrak{A})$ is a *locally compact* Hausdorff space in the Gelfand topology.

Given an Abelian Banach algebra $\mathfrak{A}$, the map

$$\begin{array}{cccc} \hat{} : & \mathfrak{A} & \rightarrow & \mathcal{C}_0(\sigma(\mathfrak{A})) \\ & a & \mapsto & \hat{a}, \end{array}$$

where

$$\begin{array}{cccc} \hat{a} \equiv ev_a : & \mathfrak{A} & \longrightarrow & \mathbb{C} \\ & \phi & \mapsto & \hat{a}(\phi) := \phi(a) \end{array}$$

is called **Gelfand transform** and the function $\hat{a} \in \mathcal{C}_0(\sigma(\mathfrak{A}))$ is said to be the Gelfand transform of $a \in \mathfrak{A}$.

Theorem 3.2.4 *The Gelfand transform is a norm-decreasing, hence continuous, homomorphism from the Abelian Banach algebra $\mathfrak{A}$ to the Abelian Banach algebra $\mathcal{C}_0(\sigma(\mathfrak{A}))$, i.e.*

$$\|\hat{a}\|_\infty \leq \|a\|.$$

If $\mathfrak{A}$ is a Banach algebra with unit, then the range of the Gelfand transform is contained in $\mathcal{C}(\sigma(\mathfrak{A}))$.

If the Gelfand transform is an isomorphism, then the Abelian Banach algebra $\mathfrak{A}$ is called **semisimple**.

Since an Abelian C^*-algebra is, in particular, an Abelian Banach algebra, all the results above still hold for such C^*-algebras.

[1]The w^* 'weak-star' topology on the dual of a Banach algebra $\mathfrak{A}$ is the topology in which a sequence of functionals $\{\tau_n\}_{n\in\mathbb{N}}$ is convergent if and only if, for every $a \in \mathfrak{A}$, the sequence of complex numbers $\{\tau_n(a)\}_{n\in\mathbb{N}}$ is convergent.

3.3 Gelfand theory of Abelian C^*-algebras

For Abelian C^*-algebras the Gelfand transform has more powerful properties than for Abelian Banach algebras, in fact it becomes an isomorphism which enables to find an identification of every Abelian abstract C^*-algebra as the C^*-algebra of complex-valued continuous functions on its spectrum, which is then the prototype of these kind of C^*-algebras.

Theorem 3.3.1 (Gelfand-Naimark) *If $\mathfrak{A}$ is an Abelian C^*-algebra with unit then the Gelfand transform $\hat{}$ is an isometric $*$-isomorphism from $\mathfrak{A}$ to $\mathcal{C}(\sigma(\mathfrak{A}))$. If $\mathfrak{A}$ doesn't have a unit, then the Gelfand transform is an isometric $*$-isomorphism from $\mathfrak{A}$ to $\mathcal{C}_0(\sigma(\mathfrak{A}))$.*

Corollary 3.3.1 *Every compact Hausdorff space X arises as the spectrum of an Abelian unital C^*-algebra, specifically $\mathcal{C}(X)$:*

$$X = \sigma(\mathcal{C}(X)).$$

The easy proof is left to the reader.

This result is very important, because it says that a compact Hausdorff space can be reconstructed from its Abelian unital C^*-algebra of continuous functions by calculating its spectrum. This is the starting point for generalizations to non-commutative topological spaces (see [30]).

Let's end this section by citing the link between these results on C^*-algebras and the Stone-Cech compactification.

Theorem 3.3.2 *If X is a Hausdorff space for which the Urysohn lemma holds, then the map*

$$\begin{array}{ccll} X & \longrightarrow & \sigma(\mathcal{C}(X)) & \\ x & \mapsto & \phi_x, & \phi_x(f) := f(x) \end{array}$$

is a homeomorphism with its image, which is dense in the compact Hausdorff space $\sigma(\mathcal{C}(X))$. This last is called the **Stone-Cech compactification** *of X.*

3.4 GNS representation of C^*-algebras

It's often useful to treat the elements of a C^*-algebra as operators on a Hilbert space, the correct way to do this is to consider a Hilbert representation, or simply representation, of the C^*-algebra.

Precisely, a **representation of a C^*-algebra** $\mathfrak{A}$ is a couple $(\mathcal{H}, \varphi)$ given by a Hilbert space $\mathcal{H}$ and a $*$-homomorphism:

$$\varphi : \mathfrak{A} \to \mathcal{B}(\mathcal{H}).$$

In particular the representation is said **faithful** if φ is injective.

In the theory of representations of C^*-algebras an important role is played by certain functionals, specifically: a **positive linear functional on** $\mathfrak{A}$ is a linear functional satisfying

$$\varphi(aa^*) \in \mathbb{R}^+ \qquad \forall a \in \mathfrak{A}.$$

The elements of the kind aa^* are called the positive elements of $\mathfrak{A}$, hence we can say that a positive functional on $\mathfrak{A}$ maps the positive elements of $\mathfrak{A}$ in positive real numbers.

In particular we define a **state of a C^*-algebra** to be a positive functional on $\mathfrak{A}$ of unit norm.

It's easy to see that the characters are states.

Theorem 3.4.1 *The positive functionals on a C^*-algebra $\mathfrak{A}$ are always continuous.*

The relation between positive linear functionals and representations of C^*-algebras is given by the GNS construction, so called in honor of Gelfand, Naimark and Segal, the mathematicians that built it, that enables to *associate in a unique way a representation to a positive functional on a C^*-algebra.*

Let's begin with this theorem.

Theorem 3.4.2 *Fixed a positive functional τ on a C^*-algebra $\mathfrak{A}$, the map*

$$\begin{array}{rcl} \mathfrak{A} \times \mathfrak{A} & \to & \mathbb{C} \\ (a, b) & \mapsto & \tau(b^* a) \end{array}$$

is a sesquilinear positive-definite form on $\mathfrak{A}$.

To get an inner product from this form induced by τ we miss only the non-degeneracy, to get it let's define the 'null set' of τ:

$$N_\tau := \{a \in \mathfrak{A} \mid \tau(a^* a) = 0\}$$

this is easily seen to be a left ideal of $\mathfrak{A}$ and so it's possible to take the quotient of $\mathfrak{A}$ on N_τ. Then the map

$$\begin{array}{rcl} \mathfrak{A}/N_\tau \times \mathfrak{A}/N_\tau & \to & \mathbb{C} \\ (a + N_\tau, b + N_\tau) & \mapsto & \tau(b^* a) \end{array}$$

is a well defined inner product on $\mathfrak{A}/N_\tau$, which can be completed to a Hilbert space which we indicate with $\mathcal{H}_\tau$.

Let's define, $\forall a \in \mathfrak{A}$, an operator $\varphi(a) \in \mathcal{B}(\mathfrak{A}/N_\tau)$ as below:

$$\varphi(a)(b + N_\tau) = ab + N_\tau.$$

This operator is bounded and so it can be extended in a unique way to a bounded operator of $\mathcal{B}(\mathcal{H}_\tau)$ denoted with $\varphi_\tau(a)$.

Finally one can prove that the map:

$$\begin{array}{rccc} \varphi_\tau : & \mathfrak{A} & \rightarrow & \mathcal{B}(\mathcal{H}_\tau) \\ & a & \mapsto & \varphi_\tau(a) \end{array}$$

is a $*$-homomorphism.

So, starting from a positive functional τ, the GNS procedure has given us two objects: a Hilbert space $\mathcal{H}_\tau$ and a $*$-homomorphism φ_τ, with these ones we can construct a representation of $\mathfrak{A}$.

Specifically we call the representation $(\mathcal{H}_\tau, \varphi_\tau)$ the **GNS representation** of $\mathfrak{A}$ associated to the positive functional τ, which is called the **generator** of the representation $(\mathcal{H}_\tau, \varphi_\tau)$.

If we take the direct sum of the GNS representations w.r.t. all the positive functionals of $\mathfrak{A}$ we get the so-called **universal representation of** $\mathfrak{A}$:

$$(\bigoplus_\tau \mathcal{H}_\tau, \bigoplus_\tau \varphi_\tau).$$

The importance of the universal representation is contained in the following result.

Theorem 3.4.3 (Gelfand-Naimark) *The universal representation of a non empty C^*-algebra is faithful.*

Thanks to the Gelfand-Naimark theorem one has that **every C^*-algebra $\mathfrak{A}$ can be identified with a C^*-subalgebra of $\mathcal{B}(\mathcal{H})$**, this is one of the reasons for the great attention reserved to the C^*-algebra of bounded linear operators on a Hilbert space.

3.5 States of a C^*-algebra and probability measures on its spectrum

There is a faithful correspondence between states of an Abelian unital C^*-algebra and probability measures on its spectrum. This correspondence relies on the well known Riesz-Markov theorem [78].

Theorem 3.5.1 (Riesz-Markov) *To every positive linear functional L on $\mathcal{C}(X)$, X compact Hausdorff space, there corresponds a unique positive regular Borel measure μ on X such that:*

$$L(f) = \int_X f \, d\mu \qquad \forall f \in \mathcal{C}(X).$$

Moreover $\| L \| = \| \mu \|$.

If we use the Gelfand isomorphism to identify $\mathfrak{A}$ with $\mathcal{C}(\sigma(\mathfrak{A}))$, then the Riesz-Markov theorem implies that *there is an isomorphism between positive linear functionals on $\mathfrak{A}$ and positive regular Borel measures on $\sigma(\mathfrak{A})$*.

The representation of the positive linear functional φ_μ associated to the positive regular Borel measure μ is given by:

$$\varphi_\mu(a) = \int_{\sigma(\mathfrak{A})} \hat{a} \, d\mu \, .$$

The isometric behavior of the Riesz-Markov isomorphism can be understood easily by observing that $\|\varphi_\mu\| = \varphi_\mu(\mathbf{1}) = \int_{\sigma(\mathfrak{A})} \mathbf{1} \, d\mu = \mu(\sigma(\mathfrak{A}))$, but μ is positive, hence $\mu(\sigma(\mathfrak{A})) = |\mu|(\sigma(\mathfrak{A})) = \|\mu\|$, thus $\|\varphi_\mu\| = \|\mu\|$.

The considerations above prove that **the states of $\mathfrak{A}$ are in bijection with the probability measures on** $\sigma(\mathfrak{A})$.

Chapter 4

Canonical loop quantum gravity

4.1 Introduction

In this chapter I will describe the mathematical framework that relies at the basis of canonical loop quantum gravity.

This quantization procedure, if compared with other ones − for example superstring theory − is very conservative, in fact no new symmetries of nature (as supersymmetry or other non yet observed symmetries) and no new structures (as strings or branes) are postulated. Instead one tries to remain as close as possible to the very basic ideological foundations of quantum mechanics and general relativity.

This means that canonical loop quantum gravity describes states as rays in a suitable Hilbert space and physical observables as self-adjoint operators on the Hilbert space of states, as quantum mechanics prescribes, and it does this in a manifest background-independent (or diffeomorphism-invariant) way, i.e. without relying on a manifold with fixed geometry, thus respecting Einstein's equivalence principle, the basic principle of general relativity.

Since quantum gravity is expected to describe effect at the Planck scale $\ell_P = \sqrt{hG/c^3} \approx 10^{-35}$ m, or 10^{19} GeV, and since with the most powerful accelerator of the world, the LHC, we can scan only distances of 10^{-13} m (namely we can produce only energies of 10^3 GeV), unfortunately there are no experimental data to fit at the Planck scale, thus the only reasonable way to construct a quantum theory of gravity is to mix the peculiarities of quantum mechanics and general relativity guided by *mathematical consistency*. This may seem 'a walk in the dark', but, even though we are far from reaching the energies of the Planck scale with accelerators, there is a hope to discover *indirect* consequences of the quantum structure of spacetime at the Planck scale, e.g. by the calculations of the black holes entropy and maybe other phenomena in the near future.

In every step of the quantization algorithm of loop quantum gravity there is a strong insistence on background-independence: this is a very big difference w.r.t. the techniques used in the standard model of electroweak and strong nuclear interactions, in fact the main tools of the standard model are the Feynman diagrams, which − for short − amount to operate some perturbations on a distinguished, externally prescribed structure: the Minkowski space! Even though the standard model describes in a spectacular way all the interactions except gravity, the manifest violation of the equivalence principle (which, mathematically speaking, is contained in the diffeomorphism invariance of Einstein's equations) is clearly unacceptable when the force of gravity becomes as important as the other forces. This is the physical reason why people working on loop quantum gravity reserve so much importance on the background-independence of the whole formalism.

Before starting to describe in detail the framework of canonical loop quantum gravity I present a short chronological account of the most important steps in the construction of this theory.

4.2 A brief history of loop quantum gravity

Two classical references for the chronological evolution of canonical loop quantum gravity are [42] and [90].

1986 Historically, the very beginning of canonical loop quantum gravity relies of the fundamental article of Ashtekar [4] of 1986, in which, as already said in the first chapter, he discovered that only the self-dual part of the connection appearing in the Palatini action is necessary to derive, with variational techniques, the Einstein equations describing gravity at a classical level. In term of Ashtekar's 'new variables' (as it was custom to say at that time) the constraints of general relativity greatly simplify, becomes closed under Poisson bracket and functionally easier than in every other formalism. This fact attracted many researchers in the attempt to canonically quantize gravity;

In the same year Gambini and Trias pointed out in a systematical way in [43] the usefulness of Wilson loop functions in the non-perturbative quantization of gauge theories. The Wilson functions appeared the first time in the physical literature in a pioneering article of Wilson [96] about quark confinement in quantum chromodynamics. Inspired by an article of Giles [44] of 1981, Gambini and Trias showed that, for Yang-Mills theories with compact gauge group, the Wilson functions capture the full gauge-invariant information of connections (we will be more rigorous and clear later about this very important argument);

1990 Rovelli e Smolin rediscovered the ideas of Gambini and Trias and applied them to construct (formal) diffeomorphism-invariant states of canonical quantum gravity with the help of a generalization of the Fourier transform called 'loop transform';

1992 This year is fundamental for loop quantum gravity. In fact Ashtekar and Isham in [6] constructed the first prototype of quantum configuration space for gravity by extending the concept of smooth connection to a distributional object called 'generalized connection'. To obtain this result they used powerful techniques of C^*-algebras theory, this was a great intuition, because in the following development of loop quantum gravity the C^*-algebraic formalism has been proved to be very helpful;

1994-1995 In these years Ashtekar, Lewandowski, Marold and Mourão managed to construct in a rigorous way a faithful probability measure on the quantum configuration space proposed by Ashtekar and Isham with the help of projective techniques: [7], [66]. This is one of the most important results on the mathematical side of loop quantum gravity, since with that measure the authors were able to endow with an inner product the quantum configuration space, thus constructing the Hilbert space of kinematical states of quantum gravity. Moreover, the measure turned out to be invariant under both gauge transformations and spatial diffeomorphisms, this lead to the construction of (rigorously defined) diffeomorphism-invariant states with the help of a (no more formal) loop transform. Finally the projective techniques were used also to endow the quantum configuration space with a differential structure, which enabled to perform differential and integral calculus on it; see [8], [9];

1995 The fundamental role of Wilson functions in a non-perturbative quantization program were firmly clear. The major problem with these functions was their non-independence (overcompleteness), because of Mandelstam identities. Rovelli and Smolin, inspired by a pioneering work of Penrose [74], were able to find out independent functionals of connections, later called spin network states. Baez then showed in [16] and [17] that spin network states are enough to form an orthonormal basis of the Hilbert space of kinematical states of quantum gravity;

1995-1996-1997 Rovelli and Smolin derived in [82] volume and area operators for quantum gravity, showing that they have a discrete spectrum, i.e. area and volume, at the Planck scale, are quantized! This outstanding result was successively confirmed by the works of Loll [62] Ashtekar and Lewandowski [10], [11].

2000 As already said, in absence of direct measurements of the quantum nature of spacetime at the Planck scale, one of the possible tests of quantum gravity is the computation of Hawking-Bekenstein black holes entropy. This calculation was done in [5] and shown to be in agreement with the known result of Hawing-Bekenstein entropy, thus giving the first indication that loop quantum gravity *can* be physically consistent.

The items above describe only some of the most important results in the construction of canonical loop quantum gravity, the quoted works correspond to the knowledge and the taste of the author. Obviously many other non-quoted works are important as well and the research in canonical quantum gravity is still under investigations in many areas.

Now we want to explore in a systematic way each of the steps presented above. Since we have already described Ashtekar's formulation of gravity, the next step is the introduction of the Wilson loop functions to develop a non-perturbative quantization of gauge theories. The researches on loop quantum gravity have shown that the structure of the loops to which the Wilson functions correspond have a dramatic consequence on the whole theory, for this reason it is worthwhile to start with a wide discussion of the various loops and loop groups available in literature.

4.3 Loops, paths and graphs embedded in a manifold

In this section $P(M,G)$ will denote a principal bundle in which the base manifold M is taken to be an ordinary manifold of dimension $dim(M) > 1$ equipped with a fixed *real analytic structure*. The choice of this structure is due to the fact that, at the time of writing, the most important results in the developments of the loop quantization necessitate the use of piecewise analytic loops in M. An investigation of the pure smooth case has been performed in [22], [23] and [60]. Anyway the relation between the smooth and the real-analytic differential structures is as nice as possible: every ordinary smooth manifold admits a real analytic structure unique up to smooth diffeomorphisms.

It is worth remembering some terminology about paths and loops: a **path** in M is a *continuous and piecewise* $\mathcal{C}^1$ map of the form:

$$\gamma : [a, b] \to M$$

$[a, b] \subset \mathbb{R}$ is called **domain of parameterization** of γ.

A **closed path** in M, i.e. a path for which $\gamma_a = \gamma_b$, is commonly called **loop** in M.

γ is said to be **analytic** on $[a, b]$ if it is the restriction of an analytic map defined on an open set containing $[a, b]$.

γ^* will indicate the image of the path γ, that is the subset of M given by $\{\gamma_t \in M \mid t \in [a, b]\}$.

An **arc** of γ is any subset $\gamma_i^* \subset \gamma^*$ such that γ_i^* is the image of an analytic restriction $\gamma_i : [t_i, t_{i+1}] \to M$ of γ.

The **composition of two paths**, say $\gamma_1 : [a, b] \to M$ and $\gamma_2 : [c, d] \to M$, is subjected to two conditions, in fact it can be defined only when $b = c$ and $\gamma_1(b) = \gamma_2(c)$; if this is the case then the composed path is $\gamma_1\gamma_2 : [a, d] \to M$, defined by

$$\gamma_1\gamma_2(t) := \begin{cases} \gamma_1(t) & t \in [a, b] \\ \gamma_2(t) & t \in [c, d] \end{cases}$$

thus $\gamma_1\gamma_2$ is simply the path obtained travelling first the path written on the left and then the path written on his right. Observe that **this convention is opposed to that commonly used to define composition of functions!** For this reason the composition of functions will be indicated by the symbol $\circ$ and the composition of paths will be denoted simply by juxtaposing the paths in the order shown above.

The composition of many paths is defined in an analogous fashion.

The **inverse path** γ^{-1} is simply the path γ travelled in the opposite direction, that is

$$\gamma^{-1} : [a, b] \to M,\ \gamma^{-1}(t) := \gamma(a + b - t)\,.$$

The set *Path* given by the whole collection of paths in M equipped with the composition law defined above is a **groupoid**, that is, roughly speaking, a semigroup with composition law not always defined. More rigorously a groupoid is defined to be a set Λ endowed with a binary operation (indicated with the juxtaposition of its elements) satisfying the following properties:

- for every element $\lambda \in \Lambda$ there exists an element $\lambda^{-1} \in \Lambda$, called its inverse, such that $r(\lambda) := \lambda\lambda^{-1}$ and $l(\lambda) := \lambda^{-1}\lambda$ exist and are the right and the left unit of λ, respectively;

- for every $\lambda, \eta \in \Lambda$, the product $\lambda\eta$ is defined if and only if $r(\eta) = l(\lambda)$;
- when defined, the product is associative.

If $\mathsf{U}(\Lambda)$ denotes the set of the units of Λ, i.e. the collection of all the elements of the form $\lambda\lambda^{-1}$, for some $\lambda \in \Lambda$, it is obvious that a groupoid Λ is a group if and only if $\mathsf{U}(\Lambda)$ is a singleton.

There is an important concept concerning the parameterization of a path: given two paths γ and η in M with domains $[a, b]$ and $[c, d]$, respectively, if there exists a *diffeomorphism* $\tau : [a, b] \to [c, d]$ such that

$$\begin{array}{ccc} M & = & M \\ {\scriptstyle\gamma}\uparrow & & \uparrow{\scriptstyle\eta} \\ [a,b] & \xrightarrow[\tau]{} & [c,d] \end{array}$$

i.e. $\gamma(t) = \eta(\tau(t))$, $\forall t \in [a, b]$, then the paths γ and η are said to be one the **reparameterization** of the other. It is obvious that two reparameterized paths have identical image, but the same point on the common image is reached for different values of the parameter t.

Since τ is a real diffeomorphism, it must be monotone and this leads to the following classification:

- if τ is a growing function then γ and η are said to have the *same orientation*;
- if τ is a decreasing function then γ and η are said to have *opposite orientation*.

It is easy to see that the relation "orientation-preserving reparameterization" is an equivalence relation in the set of all path in M, thus it is well posed the following definition.

Def. 4.3.1 *An* **oriented path in** M *is an equivalence class of paths in* M *w.r.t. the equivalence relation "orientation-preserving reparameterization".*

For oriented paths it is usual to choose as domain of parameterization the real closed interval $[0, 1]$, this choice is possible by virtue of the diffeomorphism

$$\begin{array}{ccc} [a,b] & \longrightarrow & [0,1] \\ t & \mapsto & \frac{t-a}{b-a} \,. \end{array}$$

There is a natural equivalence relation on the set of oriented paths which will be very useful in the sequel. Its definition necessitates the introduction of the concept of **immediately retraced path**.

Def. 4.3.2 *A path γ is said to be immediately retraced if it can be written as $\gamma = \prod_i \gamma_i {\gamma_i}^{-1}$, for some paths γ_i in M.*

The following equivalence relation on the set of oriented paths in M was first introduced by Chen.

Def. 4.3.3 *Two oriented path γ_1, γ_2 are said to be* **elementary equivalent***, $\gamma_1 \sim_{el} \gamma_2$, if one is obtained from the other by composition with an immediately retraced path γ, i.e. $\gamma_1 = \gamma_2\gamma$.*

It is worth noting that *in every elementary class of paths there is only one representative free from immediate retracing*, this representative is the path for which the immediately retraced path γ is the constant path in the ending point of the path itself (for all the other representatives γ is not trivial). This path is taken to be *the canonical representative* of the elementary equivalence class to which it belongs.

For the purposes of the loop quantization it will be seen that it is very important to have at one's hand the definition of finite graph embedded in a manifold, or simply graph, this needs the concepts of edge and vertex, which are introduced below.

Def. 4.3.4 *An* **edge** *in M is a continuous map $e : [0, 1] \to M$ such that its restriction $\tilde{e} \equiv e|_{(0,1)}$ is an analytic embedding*[1] *of $(0, 1)$ in M.*

The **vertexes** *of an edge are its starting and ending point, that is $e(0) \equiv s(e)$ and $e(1) \equiv t(e)$, also called* **source** *and* **target***, respectively. A vertex is called* **n-valent** *if there are n edges meeting in it. Obviously n is called the* **valence** *of the vertex itself.*

It is not possible to define the vertexes of an edge without choosing a parameterization because every representative in the class has, in general, different source and target, due to the reparameterization.

Def. 4.3.5 *A* **graph** *in M is the union of a finite family of images of edges intersecting only in their vertexes. A graph is said to be* **connected** *if the source of every edge is the target of another one.*

The usual symbol to denote a graph is Γ; the number of edges and vertexes of Γ will be indicated by E_Γ and V_Γ, respectively.

The following result is of essential importance:

[1] This means that $\tilde{e}$ is analytic and injective, with injective tangent map and $\tilde{e}^*$ is a submanifold of M w.r.t. the topology inherited by M.

Theorem 4.3.1 *For any piecewise analytic path (resp. loop) γ in M, its image γ^* is a graph (resp. connected graph) in M.*

Proof. Suppose first that γ is globally analytic, then its tangent map is injective but, at least, in a finite number of points, thus γ is a local embedding.

Moreover $[0, 1]$ is compact and γ is continuous, thus γ^* is also compact and so it can be covered by a finite number of open analytic submanifolds of M given by some opportune arcs ${\gamma_i}^*$. The easiest way to obtain these arcs is to chose a partition of $[0, 1]$ and to consider the restriction of γ to the subintervals.

Thanks to the assumption of analyticity, the arcs ${\gamma_i}^*$ intersect only in a finite number of point or they agree, hence the covering of γ^* can be refined by taking all the arcs which intersect themselves only in the extreme points.

This arcs clearly become the edges of the graph γ^* and their points of intersections become its vertexes.

If γ is only piecewise analytic, then the arguments above work again on every piece on which the path is analytic. □

In the proof of the theorem it has been shown that *the graph associated to a piecewise analytic path is not uniquely defined*, in fact it depends on the partition of $[0, 1]$ which leads to the covering of γ^*. Obviously the graph is fixed when this partition is chosen.

The result just proved will be very useful in the sequel, hence we make the following: **the paths considered in the sequel will always assumed to be piecewise analytic**. The developing of a theory which uses piecewise smooth paths, instead of the analytic ones, is still under investigation.

Now we leave the generic paths and we put our attention on the loops, which will be indicated with α, β, ... All the loops will be based on the same point $\star \in M$, unless otherwise specified.

The first fact to stress is that one can easily define a law of composition between two loops α and β in this way

$$(\alpha\beta)(t) := \begin{cases} \alpha(2t) & \text{if t} \in [0, \frac{1}{2}] \\ \beta(2t-1) & \text{if t} \in [\frac{1}{2}, 1] \end{cases}$$

but this law doesn't give to the set of loops the structure of a group, in fact the composition between a loop α and its "inverse"

$$\begin{array}{rccl} \alpha^{-1}: & [0,1] & \longrightarrow & M \\ & t & \mapsto & \alpha^{-1}(t) := \alpha(1-t) \end{array}$$

is an immediately retraced loop, but it doesn't agree with the constant loop $\star$, which is obvious to take as the unit loop. Hence the set of all loops in M

based on $\star$ with the composition law defined above is a semigroup with unit $\star$, usually denoted with $\Omega_\star(M)$, and not a group.

It is obvious that to give the group structure to the set of loops one has to take the quotient w.r.t. an equivalence relation which puts the loops of the type $\alpha\alpha^{-1}$ in the same class of the unit loop $\star$.

A natural equivalence relation to get this is the already defined elementary equivalence, in fact $\alpha\alpha^{-1} \sim_{el} \star \Leftrightarrow \exists\, \gamma$, immediately retraced path, such that $\alpha\alpha^{-1} = \star\gamma$, but such γ do exists and it is simply $\alpha\alpha^{-1}$!

Thanks to this arguments the following definition has perfectly sense.

Def. 4.3.6 *The set of all classes of elementary equivalence of loops in M based on $\star$ with composition law given by $[\alpha]_{el}[\beta]_{el} := [\alpha\beta]_{el}$ is a group called the* **group of loops** *and it is denoted by $L_\star(M)$.*

Obviously the unit of $L_\star(M)$ is $[\star]_{el}$ and the inverse of $[\alpha]_{el}$ is $[\alpha^{-1}]_{el}$. For shortness the class $[\alpha]_{el}$ will be identified with its canonical representative α.

The next step is to define the concepts of independent and simple loops.

Def. 4.3.7 *An oriented arc l of a path γ is said to be* **simple** *if $t_1 \neq t_2$ implies $l(t_1) \neq l(t_2)$, $\forall t_1, t_2 \in [0,1]$, i.e. l doesn't intersect itself (equivalently, the set $l^{-1}(x)$ is a singleton $\forall x \in l^*$).*

$\alpha \in L_\star(M)$ is said to be a **simple loop** *if in its elementary equivalence class there exists a representative which admits a simple arc.*

A finite family $\{\beta_i\}(i = 1, \ldots, n) \subset L_\star(M)$ is said to be **independent** *if every β_i admits a simple arc l_i such that $l_i \cap {\beta_j}^* = \emptyset\ \forall i \neq j$, i.e. every loop of the family has a simple arc which doesn't intersect the images of the other loops of the same family. The β_i's are said to be* **independent loops***.*

A very easy example of independent family of loops is the following: take $M \equiv \mathbb{R}^2$ and take γ_+,γ_- to be the upper and the lower half unit circle in $\mathbb{R}^2$, respectively. It is obvious that $\{\gamma_+, \gamma_-\}$ is an independent family. Instead the family $\{\gamma, \gamma_+, \gamma_-\}$, with $\gamma \equiv S^1$, is not independent, because every arc of γ intersect γ_+ or γ_- or both of them.

The importance of the notion of independence between loops is motivated by the next theorem.

Theorem 4.3.2 *Every $\alpha \in L_\star(M)$ is the composition of a finite family of simple independent loops.*

Proof. For simplicity fix the canonical representative α of an elementary class of loops in $L_\star(M)$ and a finite partition of $[0,1]$:

$$0 \leq a_1 < b_1 < \ldots < a_i < b_i < a_{i+1} < b_{i+1} \ldots < a_m < b_m \leq 1$$

such that $\alpha|_{[a_i,b_i]} \equiv l_i$ is an edge of the graph α^* and *every* edge of α^* is obtained in this way (we know that such a partition exists by the previous theorem).

For any fixed edge l_i, $1 < i < m$, take:

- a real analytic path ${q^i}_-$ connecting $\star$ with $\alpha(a_i)$, the source of l_i;
- a real analytic path ${q^i}_+$ connecting $\star$ with $\alpha(b_i)$, the target of l_i.

Take also ${q^1}_- \equiv \star$, ${q^m}_+ \equiv \star$ and ${q^{i-1}}_+ \equiv {q^i}_-$, $i = 2, \ldots, m$.

Then the collection $\{\beta_{l_i} := {q^i}_- l_i ({q^i}_+)^{-1}\}(i = 1, \ldots, m)$ is a finite family of simple independent loops, in fact every arc properly contained in each l_i doesn't intersect any other edge l_j, $j \neq i$, by definition of edge.

For every fixed i one has

$$\cdots \beta_{l_{i-1}} \beta_{l_i} \beta_{l_{i+1}} \cdots = \star \cdots l_{i-1} ({q^i}_-)^{-1} {q^i}_- l_i ({q^{i+1}}_-)^{-1} {q^{i+1}}_- l_{i+1} \cdots \star$$

hence the composition of powers of the loops β_{l_i} reconstructs α up to immediately retraced loops, i.e. it belongs to the same elementary equivalence class of α and so the theorem is proved. □

An easy, but useful, generalization of the last theorem is the following.

Corollary 4.3.1 *Given a finite family $\{\alpha_1, \ldots, \alpha_r\} \subset L_\star(M)$, every loop of the family can be written as composition of loops belonging to a simple independent family $\{\beta_1, \ldots, \beta_s\} \subset L_\star(M)$, for a certain integer $s \geq r$.*

Proof. To every loop α_j associate a graph Γ_j with edges $\{l_{j_i}\}$ with the property that the edges of two different graphs intersect themselves only in their vertexes or they coincide (we know that this is always possible by refining the parameterization of the loops). Now for every loop α_j define the simple independent loops β_{j_i} as in the proof of the theorem above, then the theorem follows from the same considerations. □

By iterating the arguments of the proofs above one has that if the family $\{\alpha_1, \ldots, \alpha_r\} \subset L_\star(M)$ is extended to a family $\{\alpha_1, \ldots, \alpha_r, \alpha_{r+1}, \ldots, \alpha_{r'}\} \subset L_\star(M)$ then there is another (in general different) family of simple independent loops $\{\beta'_1, \ldots, \beta'_{s'}\} \subset L_\star(M)$ which decomposes every loop of the extended family and every loop of the independent family $\beta_1, \ldots, \beta_s$.

Fixed a generic $\alpha \in L_\star(M)$, the more general decomposition of α as a product of powers of independent loops $\beta_1, \ldots, \beta_m$ can be written as:

$$\alpha = {\beta_1}^{n_{1,1}} \cdots {\beta_m}^{n_{m,1}} {\beta_1}^{n_{1,2}} \cdots {\beta_m}^{n_{m,2}} \cdots {\beta_1}^{n_{1,k}} \cdots {\beta_m}^{n_{m,k}}$$

where $n_{i,j} \in \mathbb{Z}$, for every $i = 1, \dots, m$, $j = 1, \dots, k$. This decomposition will be used many times in the sequel.

If the elementary equivalence is the most natural, there are two other important equivalence relations between loops:

1. the *thin equivalence*;
2. the *holonomic equivalence.*

The first has a topological nature while the second has a geometrical character.

The quotient of the group of loops w.r.t. these relations gives rise to other groups and the structural relations between these groups are very interesting and useful.

The definition of thin equivalence relies on the concept of **thin loop**.

Def. 4.3.8 *A loop $\alpha \in L_\star(M)$ is said to be thin if it is homotopic to the constant loop $\star$ with a homotopy having image entirely contained in α^*.*

The definition is well posed because the immediately retraced loops are obviously thin, hence there is independence from the particular choice of the representative in the class $[\alpha]_{el}$.

Def. 4.3.9 *α,$\beta \in L_\star(M)$ are said to be* **thin equivalent**, *$\alpha \sim_{th} \beta$, if there exists a thin loop γ such that $\alpha = \beta\gamma$.*

The set of all thin loops is easily recognized to be a normal subgroup of $L_\star(M)$ and so it is defined the quotient group $\mathcal{L}_\star(M) := L_\star(M)/Thin_\star(M) = \{\beta\gamma \mid \gamma \text{ thin}, \beta \in L_\star(M)\}$.

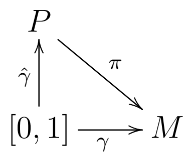
The definition of holonomic equivalence of loops requires the geometrical notion of holonomy, to which is entirely dedicated the following section.

4.4 Holonomy and holonomic equivalence of loops

In this section are presented the definitions and results which lead to the fundamental concept of holonomy. This one is then used to define the holonomic equivalence of loops.

A **horizontal lift** of a path $\gamma : [0,1] \to M$ is a path $\hat{\gamma} : [0,1] \to P$ satisfying the following conditions:

1. **lift condition**: the following diagram is commutative:



   i.e. $\pi(\hat{\gamma}_t) = \gamma_t$, for every $t \in [0,1]$;

2. **horizontal condition**: $\dot{\hat{\gamma}}_t \in H_{\gamma_t}(P)$, for every $t \in [0,1]$.

The notion of horizontal lift of a curve is closely related to that of a vector field, in fact if the vector field $\hat{X}$ on P is the horizontal lift of a vector field X on M, i.e. $\pi_* \hat{X} = X$, then the integral curve of $\hat{X}$ which starts in $p_0 \in P$ is the horizontal lift of the integral curve of X which starts in $\pi(p_0)$.

The most remarkable fact about horizontal lift of paths is expressed in the next theorem.

Theorem 4.4.1 *Let (P, M, G, π, R) be a principal fiber bundle and let γ be a smooth path in M. Then, fixed an arbitrary point $p_0 \in \pi^{-1}(\gamma_0)$, there exists one and only one horizontal lift $\hat{\gamma}$ of γ which starts in p_0.*

Proof. Let us start with the hypothesis that the bundle is trivial, then $P = M \times G$ and a lift of γ which starts in p_0 is suddenly individuated in the path $\eta_t := (\gamma_t, e)$. In fact, thanks to the global trivialization, one has $\pi(\eta_t) = pr_1(\gamma_t, e) = \gamma_t$, furthermore $\eta_0 = (\gamma_0, e) = p_0$.

If $\hat{\gamma}$ is another lift of γ which starts from p_0, then the lift condition implies that $\hat{\gamma}_t, \eta_t \in \pi^{-1}(\gamma_t)$, for every $t \in [0,1]$, hence, since the action of G on P is free and transitive on the fibers, $\hat{\gamma}$ must be of the form $\hat{\gamma}_t = \eta_t.g_t$, $\forall t \in [0,1]$, where $t \to g_t$ is a path in G starting from e. This initial condition is due to the fact that $\hat{\gamma}_0 = p_0 = \eta_0.g_0$.

Observe now that $\hat{\gamma}_t = \eta_t.g_t \equiv R(\eta_t, g_t)$, $\forall t \in [0,1]$, thus one can consider $\hat{\gamma}$ as the following composition of maps:

$$\begin{array}{ccccc} [0,1] & \longrightarrow & P \times G & \longrightarrow & P \\ t & \overset{(\eta\times g)}{\mapsto} & (\eta_t, g_t) & \overset{R}{\mapsto} & \eta_t.g_t \end{array}$$

so that $\hat{\gamma} = R \circ (\eta \times g)$.

The generalized Leibnitz rule enables to decompose the push-forward of this map as follows:

$$(R \circ (\eta \times g))_*(\dot{\eta}_t, \dot{g}_t) = (R_{g_t})_*(\dot{\eta}_t) + (R_{\eta_t})_*(\dot{g}_t).$$

Now, $\hat{\gamma}$ is a horizontal lift of γ if and only if $A(\hat{\gamma}_*(\dot{\eta}_t, \dot{g}_t)) = 0$, for every $t \in [0,1]$, i.e. if and only if

$$0 = A((R_{g_t})_*(\dot{\eta}_t)) + A((R_{\eta_t})_*(\dot{g}_t)),$$

but, thanks to the equivariance of A, the first term on the right hand side is precisely $Ad_{g_t^{-1}} A(\dot{\eta}_t)$.

Even the second term can be re-written in a more useful way, the functorial property of the push-forward implies $(R_{\eta_t})_* = (R_{\eta_t \cdot g_t})_* \circ (L_{g_t^{-1}})_*$, and so

$$A((R_{\eta_t})_*(\dot{g}_t)) = A((R_{\eta_t \cdot g_t})_*(L_{g_t^{-1}})_*(\dot{g}_t)) = (L_{g_t^{-1}})_*(\dot{g}_t)$$

where the last equality follows from the property of A to reproduce the generators of the fundamental vector fields.

The quantity $(L_{g_t^{-1}})_*(\dot{g}_t)$ is called **logarithmic derivative** and the mapping $t \mapsto (L_{g_t^{-1}})_*(\dot{g}_t)$ is a curve in $\mathfrak{g}$ because $(L_{g_t^{-1}})_*(\dot{g}_t) \in T_{g_t^{-1}g_t}G = T_eG \simeq \mathfrak{g}$.

The horizontal condition for the lift $\hat{\gamma}$ can now be re-written as

$$0 = Ad_{g_t^{-1}} A(\dot{\eta}_t) + (L_{g_t^{-1}})_*(\dot{g}_t)$$

or, more explicitly, using the fact that G is supposed to be a matrix group,

$$0 = {g_t}^{-1} A(\dot{\eta}_t) g_t + {g_t}^{-1} \dot{g}_t$$

i.e.

$$\dot{g}_t = -A(\dot{\eta}_t) g_t$$

which is a non-autonomous first-order ordinary linear differential equation in G, written in the normal form.

The non-autonomy of the equation depends on the fact that $A(\dot{\eta}_t)$ is an explicitly t-dependent (left-invariant) vector field on G.

Since the path $t \mapsto g_t$ must satisfy the initial condition $g_0 = e$, the horizontal condition for γ is actually equivalent to this Cauchy problem:

$$\begin{cases} \dot{g}_t = -A(\dot{\eta}_t) g_t \\ g_0 = e \end{cases}$$

hence the theorem of existence and uniqueness guarantees that this problem admits a unique solution.

If (P, M, G, π, R) is not trivial, then it is locally trivial, hence, fixed a gauge by the choice of a local section σ_α, an obvious lift of γ starting from p_0 is $\hat{\gamma} := \sigma_\alpha \circ \gamma$, in fact, by definition of section, $\pi(\hat{\gamma}_t) = \pi(\sigma_\alpha(\gamma_t)) = id_{U_\alpha}(\gamma_t) = \gamma_t$, for every $t \in [0,1]$.

It follows that $\dot{\hat{\gamma}}_t = (\sigma_\alpha)_*(\dot{\gamma}_t)$ and so $\hat{\gamma}_t$ is horizontal if and only if $A(\dot{\hat{\gamma}}_t) = A((\sigma_\alpha)_*(\dot{\gamma}_t)) = \sigma_\alpha{}^*A(\dot{\gamma}_t) = A_\alpha(\dot{\gamma}_t) = 0$.

Hence the differential equation reached in the trivial case projects on U_α to give the differential equation:

$$\dot{g}_t = -A_\alpha(\dot{\gamma}_t)g_t.$$

When the image of the path γ is not contained in a single fibered chart U_α then it necessitates to consider a collection of fibered charts which cover the entire image of γ and to operate the same construction as above.

The solutions of the differential equations one reaches fit smoothly in the intersection of the charts, this is a consequence of the already cited transformation rule of the local connections for different choice of gauge, i.e.

$$A_\alpha = Ad_{g_{\alpha\beta}(x)^{-1}}A_\beta + g_{\alpha\beta}(x)^{-1}(g_{\alpha\beta})_* \qquad \forall x \in U_\alpha \cap U_\beta$$

from which it follows that $A_\alpha(\dot{\gamma}_t) = 0$, i.e $\dot{g}_t = -A_\alpha(\dot{\gamma}_t)g_t$, if and only if $0 = Ad_{g_{\beta\alpha}(x)^{-1}}A_\beta(\dot{\gamma}_t) + g_{\beta\alpha}(x)^{-1}(g_{\alpha\beta})_*$, $\forall x \in U_\alpha \cap U_\beta$, and so, in particular, this relation holds when $g_{\alpha\beta}(x) = g_t$, for every value of the parameter t such that $\gamma_t \in U_\alpha \cap U_\beta$, thus

$$0 = g_t{}^{-1}A_\beta(\dot{\gamma}_t)g_t + g_t{}^{-1}\dot{g}_t$$

i.e. $\dot{g}_t = -A_\beta(\dot{\gamma}_t)g_t$. In conclusion, the two differential equations are the same, or, equivalently, the left-invariant vector fields $A_\alpha(\dot{\gamma}_t)$ and $A_\beta(\dot{\gamma}_t)$ are the same for every t such that $\gamma_t \in U_\alpha \cap U_\beta$, which was the last thing to proof. □

4.4.1 Parallel transport in a principal fiber bundle

By varying the point p_0 in the fiber $\pi^{-1}(\gamma_0)$, one obtains a map from the fiber $\pi^{-1}(\gamma_0)$ to the fiber $\pi^{-1}(\gamma_1)$, defined obviously by

$$\begin{array}{rccc} \wp_{\gamma,A}: & \pi^{-1}(\gamma_0) & \longrightarrow & \pi^{-1}(\gamma_1) \\ & p & \mapsto & \wp_{\gamma,A}(p) := \hat{\gamma}_1 \end{array}$$

being $\hat{\gamma}$ the unique horizontal lift of γ which starts in p.

The map $\wp_{\gamma,A}$ is called **the parallel transport relative to the connection A along the path** γ and it depends both on A and γ.

The most important properties of the parallel transport are listed below.

Theorem 4.4.2 *Let A be a principal connection on a principal fiber bundle (P, M, G, π, R) and let γ be a smooth path in M, then the parallel transport induced by A along γ has the following properties:*

1. *$\wp_{\gamma,A}$ is unaffected by orientation preserving reparameterizations of γ;*

2. *$\wp_{\gamma,A}$ is equivariant, i.e. it commutes with R_g for every $g \in G$:*

$$\wp_{\gamma,A} \circ R_g = R_g \circ \wp_{\gamma,A}, \qquad \forall g \in G$$

more explicitly

$$\wp_{\gamma,A}(p.g) = \wp_{\gamma,A}(p).g \qquad \forall p \in P, \, \forall g \in G;$$

3. *$\wp_{\gamma,A}$ is a diffeomorphism of fibers and its inverse is given by the parallel transport induced by A along γ^{-1}:*

$${\wp_{\gamma,A}}^{-1} = \wp_{\gamma^{-1},A};$$

4. *whenever the composite path $\gamma\eta$ is defined, $\wp_{\gamma\eta,A} = \wp_{\eta,A} \circ \wp_{\gamma,A}$ (the reason for the inversion of the order of γ and η is the opposite convention to compose paths and maps).*

Proof.
1. This follows from the fact that in the proof of the existence and uniqueness of the horizontal lift $\hat{\gamma}$ of γ starting from a fixed point only the direction of $\dot{\gamma}_t$, as tangent vector, has been used;

2. The equivariance of the parallel transport follows from this general feature of horizontal lifts:

Lemma 4.4.1 *If $\hat{\gamma'}$ and $\hat{\gamma''}$ are two arbitrary horizontal lifts of $\gamma : [0, 1] \to M$, then it exists a fixed $g \in G$ such that*

$$\hat{\gamma''_t} = \hat{\gamma'_t}.g,$$

in particular, if γ' and γ'' starts in the same point, then $g \equiv e$.

Proof. In the proof of the theorem 4.4.1, η was an arbitrary lift of γ which started in the same point of the horizontal lift $\hat{\gamma}$, now, instead, $\hat{\gamma'}$ and $\hat{\gamma''}$ are both horizontal lifts of γ, but they doesn't necessary start in the same point.

By the way, the lift condition imposes again that there must be a path $t \mapsto g_t$ in G such that $\hat{\gamma}''_t = \hat{\gamma}'_t.g_t$, $\forall t \in [0, 1]$, hence to prove the thesis of the lemma it suffices to prove that $t \mapsto g_t$ is a constant map.

But this is very easy, in fact by using the generalized Leibnitz rule to the identity $\hat{\gamma}''_t = \hat{\gamma}'_t.g_t$ and applying the connection A to both members one gets:

$$A(\dot{\hat{\gamma}}''_t) = Ad_{g_t^{-1}} A(\dot{\hat{\gamma}}'_t) + (L_{g_t^{-1}})_*(\dot{g}_t)$$

but $\hat{\gamma}'$ and $\hat{\gamma}''$ are both horizontal, thus $A(\dot{\hat{\gamma}}''_t) = A(\dot{\hat{\gamma}}'_t) = 0$, for every $t \in [0,1]$, hence

$$(L_{g_t^{-1}})_*(\dot{g}_t) = 0.$$

Now, $(L_{g_t^{-1}})_*$ is a linear isomorphism and so it vanishes only on the null tangent vector, i.e. $\dot{g}_t = 0$ $\forall t \in [0,1]$ and so $t \mapsto g_t$ is a constant path in G. $\square$

Thanks to this result the proof of the equivariance of the parallel transport is very easy, in fact it suffices to specialize $\hat{\gamma}'$ to be the horizontal lift of γ which starts in $p_0 \in \pi^{-1}(\gamma_0)$ and $\hat{\gamma}''$ to be the horizontal lift of γ which starts in $p_0.g \in \pi^{-1}(\gamma_0)$, then

$$p_0.g = \hat{\gamma}''_0 = \hat{\gamma}'_0.g = p_0.g$$

and, thanks to the lemma, $\hat{\gamma}''_1 = \hat{\gamma}'_1.g$ (the *same* g).

By definition $\wp_{\gamma,A}(p_0.g) = \hat{\gamma}''_1 = \hat{\gamma}'_1.g = \wp_{\gamma,A}.g$.

3. First of all it has to be proved that $\wp_{\gamma,A}$ is a bijection with inverse given by $\wp_{\gamma^{-1},A}$. This is trivial, in fact $\hat{\gamma}_0^{-1} = \hat{\gamma}_1$ and $\hat{\gamma}_1^{-1} = \hat{\gamma}_0$ thus

$$\wp_{\gamma^{-1},A}(\wp_{\gamma,A}(p_0)) = \wp_{\gamma^{-1},A}(\hat{\gamma}_1) = \wp_{\gamma^{-1},A}(\hat{\gamma}_0^{-1}) = \hat{\gamma}_1^{-1} = \hat{\gamma}_0 = p_0$$

where in the third passage it has been used the definition of the parallel transport along γ^{-1}.

Analogously one proves that $\wp_{\gamma,A}(\wp_{\gamma^{-1},A}(p_0)) = p_0$.

This shows that $\wp_{\gamma,A}$ is a bijection between fibers, furthermore it is constructed by smooth horizontal lifts of smooth curve and so it is itself smooth with smooth inverse, i.e. a diffeomorphism.

4. As a consequence of our definition of composition of paths, $\widehat{\gamma\eta}_0 = \hat{\gamma}_0 \equiv p_0 \in \pi^{-1}(\gamma_0)$, $\widehat{\gamma\eta}_1 = \hat{\eta}_1 \in \pi^{-1}(\eta_1)$, $\gamma_1 = \eta_0$ and $\hat{\gamma}_1 = \hat{\eta}_0$, hence, by definition, $\wp_{\gamma\eta,A}(p_0) = \hat{\eta}_1$.

Furthermore, $\wp_{\eta,A}(\wp_{\gamma,A}(p_0)) = \wp_{\eta,A}(\hat{\gamma}_1) = \wp_{\eta,A}(\hat{\eta}_0) := \hat{\eta}_1$.

Thus $\wp_{\gamma\eta,A}(p_0) = \hat{\eta}_1 = \wp_{\eta,A}(\wp_{\gamma,A}(p_0))$, for every fixed $p_0 \in \pi^{-1}(\gamma_0)$. $\square$

4.4.2 Holonomy and holonomy groups

If, instead of generic paths, one fixes the attention on loops, always denoted as α or β in the sequel, then the concept of holonomy arises naturally.

In fact, for a loop $\alpha \in \Omega_\star(M)$, the starting and the ending point are both $\star$, thus the parallel transport $\wp_{\alpha,A}$ is an automorphism of the fiber $\pi^{-1}(\star)$.

By varying the loop α in $\Omega_\star(M)$ one gets the set

$$\mathcal{H}_\star := \{\wp_{\alpha,A} \mid \alpha \in \Omega_\star(M)\}$$

which, thanks to the properties of the parallel transport examined above, becomes a group when it is endowed with the composition law

$$\wp_{\alpha,A} \circ \wp_{\beta,A} = \wp_{\beta\alpha,A}$$

so that ${\wp_{\alpha,A}}^{-1} = \wp_{\alpha^{-1},A}$ and the unity is $\wp_{\star,A}$.

The group $\mathcal{H}_\star$ is called **the holonomy group of the connection A in the point $\star$.**

If $\Omega^0_\star(M)$ denotes the set of the loops homotopic to the constant loop $\star$ in M, then the subgroup of $\mathcal{H}_\star$ defined by

$$\mathcal{H}^0_\star := \{\wp_{\alpha,A} \mid \alpha \in \Omega^0_\star(M)\}$$

is called the **restricted holonomy group of the connection A in the point $\star$.**

Obviously, if M is simply connected then $\mathcal{H}_\star \equiv \mathcal{H}^0_\star$.

The following discussion will be focused only on the holonomy group, analogous results holds even for the restricted holonomy groups.

The first remarkable fact about the holonomy group is that it can be conveniently represented as a subgroup of the structural group G.

Theorem 4.4.3 *Let $\mathcal{H}_\star$ be the holonomy group of a connection A on a principal fiber bundle $P(M,G)$ in the point $\star \in M$. Then, fixed a point $p \in \pi^{-1}(\star)$, the map*

$$\begin{array}{rccl} j_p: & \mathcal{H}_\star & \longrightarrow & \mathcal{H}_p \subset G \\ & \wp_{\alpha,A} & \mapsto & j_p(\wp_{\alpha,A}) := H_A(\alpha) \end{array}$$

with $H_A(\alpha)$ defined by the equation

$$\wp_{\alpha,A}(p) := p.H_A(\alpha)^{-1}$$

is a group isomorphism.

Proof. Since $\wp_{\alpha,A}$ is a diffeomorphism of $\pi^{-1}(\star)$ into itself and since $\pi^{-1}(\star)$ is a homogeneous space for G, then it exists one and only one $g \in G$ such that $\wp_{\alpha,A}(p) = p.g$, in the thesis of the theorem this g is indicated by $H_A(\alpha)^{-1}$.

This observation proofs the bijective nature of j_p and so to prove the theorem it remains only to show that j_p preserves the group structures of $\mathcal{H}_\star$ and G, i.e. that

$$j_p(\wp_{\alpha\beta,A}) = H_A(\alpha)H_A(\beta).$$

This is very easy, in fact

$$\begin{aligned}
\wp_{\alpha\beta,A}(p) &= (\wp_{\beta,A} \circ \wp_{\alpha,A})(p) = \wp_{\beta,A}(\wp_{\alpha,A}(p)) \\
&= \wp_{\beta,A}(p.H_A(\alpha)^{-1}) = \wp_{\beta,A}(R_{H_A(\alpha)^{-1}}(p)) \\
&= (\wp_{\beta,A} \circ R_{H_A(\alpha)^{-1}})(p) = (\text{equivariance of } \wp_{\beta,A}) \\
&= (R_{H_A(\alpha)^{-1}} \circ \wp_{\beta,A})(p) = R_{H_A(\alpha)^{-1}}(\wp_{\beta,A}(p)) \\
&= R_{H_A(\alpha)^{-1}}(p.H_A(\beta)^{-1}) = (R_{H_A(\alpha)^{-1}} \circ R_{H_A(\beta)^{-1}})(p) \\
&= R_{H_A(\alpha)^{-1}H_A(\beta)^{-1}}(p) = R_{(H_A(\beta)H_A(\alpha))^{-1}}(p) \\
&= p.(H_A(\alpha)H_A(\beta))^{-1}
\end{aligned}$$

□

From the proof of this last theorem it is obvious that the inversion of $H_A(\alpha)$ in the relation $\wp_{\alpha,A}(p) := p.H_A(\alpha)^{-1}$ is essential in order to make j_p into a isomorphism of groups, if one doesn't invert $H_A(\alpha)$ in the above relation then j_p becomes an anti-isomorphism of groups.

The image of j_p in G is called the **holonomy group of the connection** A **in the point** $p \in P$ and it is denoted by $\mathcal{H}_p$, in particular the element $H_A(\alpha) \in G$ is called the **holonomy of** A **with respect to the loop** α.

Alternatively on can define $\mathcal{H}_\star$ to be the image of the map which sends every loop α in the holonomy of A with respect to it, i.e.

$$\begin{array}{rccc} H_A: & \Omega_\star(M) & \longrightarrow & G \\ & \alpha & \mapsto & H_A(\alpha). \end{array}$$

Even though $H_A(\alpha\beta) = H_A(\alpha)H_A(\beta)$, for every couple of loops α and β, H_A is not a group homomorphism because $\Omega_\star(M)$ is not a group! It becomes a group homomorphism when $\Omega_\star(M)$ is quotiented with respect to an equivalence relation which doesn't affect the parallel transport. This topic will be discussed later in this chapter.

Note also that the map

$$\begin{array}{rccc} \wp_A: & \Omega_\star(M) & \longrightarrow & \mathcal{H}_\star \\ & \alpha & \mapsto & \wp_{\alpha,A} \end{array}$$

is the only one which makes the following diagram

$$\begin{array}{ccc} \Omega_\star(M) & & \\ {\scriptstyle \wp_A}\downarrow & \searrow^{H_A} & \\ \mathcal{H}_\star & \xrightarrow[j_p]{} & \mathcal{H}_p \end{array}$$

commutative.

4.4.3 Computation of holonomies

Remember the holonomies born as solutions in the ending value of the parameter $t = 1$ of the differential equation

$$\dot{g}_t = A(\dot{\alpha}_t)g_t$$

i.e. $H_A(\alpha) = g_1$ (and not g^{-1} because we have changed the sign to the equation!). Here A is identified with one of its local expressions to have a clearer notation.

It has already been remarked that the differential equation above is a non-autonomous first-order linear differential equation, luckily there is a well established theory to solve this kind of equations, in the sequel are presented the most important features of this theory.

Let E be a Banach space with norm $\| \ \|$ and let $A \in \mathcal{B}(E)$, where $\mathcal{B}(E)$ denotes the algebra of the bounded linear operators from E into itself. By virtue of the inequality $\|A^n\| \leq \|A\|^n$, for every $n \in \mathbb{N}$, the series

$$\sum_{n=0}^{+\infty} \frac{A^n}{n!}$$

is absolutely convergent in $\mathcal{B}(E)$ and thus defines a bounded linear operator on E which is called the exponential of A:

$$e^A := \sum_{n=0}^{+\infty} \frac{A^n}{n!}$$

because it agrees with the usual exponential when $E \equiv \mathbb{R}$ or $\mathbb{C}$.

A well known property of the exponential of A is that, as the ordinary real exponential, it satisfies:

$$\frac{d}{dt}(e^{tA}) = Ae^{tA} \qquad \forall t \in [0, 1]$$

hence the solution of the Cauchy problem for the curve $\xi : [0,1] \to E$ defined by

$$\begin{cases} \frac{d}{dt}(\xi_t) = A(\xi_t) & t \in [0,1] \\ \xi_0 = x \end{cases}$$

is given by $\xi_t = e^{tA}x$, $\forall t \in [0,1]$.

Since the operator A is fixed in $\mathcal{B}(E)$, the differential equation appearing in the Cauchy problem above is **autonomous**.

If one consider instead of a fixed $A \in \mathcal{B}(E)$ an operator-valued curve, denoted again with A for simplicity, $A : [0,1] \to \mathcal{B}(E)$, then the equation

$$\frac{d}{dt}(\xi_t) = A_t \xi_t \quad t \in [0,1]$$

becomes non autonomous and its solution is no more given by the exponential of A but by the so-called **temporary-ordered exponential** or **path-ordered exponential** of the operator-valued curve A.

To easily define this object it's convenient to put the attention of the case in which $t \mapsto A_t$ is a step function, i.e. there exists a finite partition $0 = t_0 < \ldots < t_n = 1$ of $[0,1]$ such that $A|_{(t_{i-1},t_i)}$ is constant $\forall i = 1, \ldots, n$, i.e., $\forall t \in (t_{i-1}, t_i)$ $A_t \equiv A^i$, fixed operator of $\mathcal{B}(E)$, for every $i = 1, \ldots, n$.

By defining the norm of the step function A as

$$\|A\| := \sup_{i=1,\ldots,n} \|A^i\|$$

it can be proved that the completion of the space of these step function with respect to the topology induced by the norm above, the so-called **space of the regulated functions**, contains in particular all the continuous (and hence all the smooth) curves from $[0,1]$ to $\mathcal{B}(E)$.

Thus if one defines the temporary-ordered exponential for the step functions, then the definition can easily be extended to all the continuous or smooth maps from $[0,1]$ to $\mathcal{B}(E)$ via uniform limit.

Now, given a step function $A : [0,1] \to \mathcal{B}(E)$, its temporary-ordered exponential is the operator of $\mathcal{B}(E)$ defined as follows:

$$T \exp \int_0^1 A_t dt := e^{\triangle t_n A^n} \cdots e^{\triangle t_1 A^1}$$

where $\triangle t_i := t_i - t_{i-1}$, $i = 1, \ldots, n$.

Observe that every factor which appears on the right hand side is the usual exponential of operators, since A^i is a fixed operator in the interval (t_{i-1}, t_i).

The reason for the name 'temporary-ordered exponential' is obviously due to the fact that the first operator which acts (i.e. the one at the right) is that relative to the smallest value of the parameter t, the subsequent operators follows by increasing values of t, **chronologically**.

The inequality

$$\|T\exp\int_0^1 A_t dt - T\exp\int_0^1 B_t dt\| \le e^{\max\{\|A\|,\|B\|\}}\|A-B\|$$

holds whenever A and B are step functions from $[0,1]$ to $\mathcal{B}(E)$, hence the mapping $A \to T\exp\int_0^1 A_t dt$ is uniformly continuous on every bounded subset of step functions and so it is uniquely extendible to the space of the regulated functions from $[0,1]$ to $\mathcal{B}(E)$ via uniform limit: if $\{A_n\}(n \in \mathbb{N})$ is a sequence of step functions uniformly convergent to the continuous (or smooth) operator-valued curve A, then

$$T\exp\int_0^1 A_t dt := \lim_{n\to+\infty} T\exp\int_0^1 A_n(t)dt.$$

The most important properties of the $T\exp$ are listed below:

1. $\left(T\exp\int_0^1 A_t dt\right)^{-1} = T\exp\int_1^0 A_t dt$;

2. $\left(T\exp\int_s^1 A_t dt\right)\left(T\exp\int_0^s A_t dt\right) = T\exp\int_0^1 A_t dt$, observe the order, consistent with the chronological growth of t;

3. if $t \mapsto A_t$ is a continuous curve in $\mathcal{B}(E)$, then the Cauchy problem for the path $\xi : [0,1] \to E$ given by

$$\begin{cases} \dot{\xi}_t = A_t(\xi_t) \\ \xi_0 = x \end{cases}$$

is solved in a unique way by $\xi_t = \left(T\exp\int_0^t A_s ds\right)x$, for every $t \in [0,1]$.

Finally let us specialize this dissertation to the computation of holonomies: the Cauchy problem for the curve $t \mapsto g_t$ given by

$$\begin{cases} \dot{g}_t = (A(\dot{\alpha}_t))g_t \\ g_0 = I \end{cases}$$

(where now $t \mapsto A(\dot{\alpha}_t)$ is a curve in $\mathfrak{g}$), is solved in a unique way by

$$g_t = \left(T\exp\int_0^t A(\dot{\alpha}_s)ds\right)I$$

and so the holonomy of A along the loop α can be written as

$$g_1 = H_A(\alpha) = T\exp \int_0^1 A(\dot{\alpha_s})ds$$

where we have omitted I, the identity matrix. By expanding the operator exponentials appearing in the expression of $T\exp$ in power series, one reaches a formula which is very useful for approximated calculus of the holonomies, often used in gauge theory on lattice:

$$\begin{aligned} H_A(\alpha) = \; & I + \int_0^1 A_{t_1} dt_1 + \int_0^1 dt_1 \int_0^{t_1} A_{t_2} A_{t_1} dt_2 + \ldots + \\ & + \int_0^1 dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} A_{t_n} \cdots A_{t_1} dt_n + \ldots , \end{aligned}$$

or, equivalently:

$$H_A(\alpha) = \sum_{n=0}^{+\infty} \int_{0 \le t_1 \le \ldots \le t_n \le 1} A_{t_n} \cdots A_{t_1} dt_1 \cdots dt_n .$$

In the relativistic theories can be misleading to interpret t as the time-parameter, hence the time-ordered exponential in such theories is more precisely called **path-ordered exponential** and it is indicated with $\mathcal{P}\exp$, so that the holonomy $H_A(\alpha)$ is denoted with

$$H_A(\alpha) = \mathcal{P}\exp \oint_\alpha A.$$

4.4.4 The holonomy map

From the properties 1. and 3. of the theorem 4.4.2 it follows that the holonomy of every loop belonging to the same elementary equivalence class is the same, so that is well defined the map which assigns to $\alpha \in L_\star(M)$ its holonomy w.r.t. a fixed connection A, i.e.:

$$\begin{aligned} H_A : \; & L_\star(M) \longrightarrow G \\ & \alpha \mapsto H_A(\alpha) \end{aligned}$$

this is called **holonomy map**.

From the property 3. and 4. of the theorem 4.4.2 one gets the following properties of the holonomy map:

1. $H_A(\alpha\beta) = H_A(\alpha)H_A(\beta)$ "**factorization property**" ;
2. $H_A(\alpha^{-1}) = H_A(\alpha)^{-1}$.

Furthermore if A is a flat connection then $H_A(\alpha) = e$, for every $\alpha \in L_\star(M)$.

From the factorization property it follows that **the holonomy map is a homomorphism from the group of loops to the gauge group**.

Thus, for **unitary gauge theories**, where the gauge group G is a subgroup of the unitary group $U(N)$, H_A **realizes an unitary representation of** $L_\star(M)$.

Now the third equivalence relation on the set of loops can be introduced.

Def. 4.4.1 *Two loops α and β are said to be* **holonomy equivalent** *if they have the same holonomy w.r.t. every connection, i.e.*

$$H_A(\alpha) = H_A(\beta) \qquad \forall A \in \mathcal{A}\,.$$

The holonomic equivalence will be indicated by $\alpha \sim_{hol} \beta$ and a class of holonomic equivalence of loops will be called **hoop**.

The quotient of $L_\star(M)$ w.r.t. the holonomic equivalence gives rise to a group called the **hoop group** *and indicated with*

$$\mathcal{H}_\star(M, G) \equiv L_\star(M)/\sim_{hol}\ .$$

The term "hoop" is the abbreviation of "holonomic equivalence class of loops". By definition, to every representative of a given hoop there corresponds the same holonomy, hence the holonomy map factorizes to a homomorphism between $\mathcal{H}_\star(M, G)$ and G by the position: $H_A|_{\mathcal{H}_\star(M,G)}([\alpha]_{hol}) := H_A(\alpha)$, for an arbitrary representative α in the hoop $[\alpha]_{hol}$.

For simplicity, the notation $[\alpha]_{hol}$ will be abbreviated by α.

The relations between the three groups $L_\star(M)$, $\mathcal{L}_\star(M)$ and $\mathcal{H}_\star(M, G)$ is based on the simple observation that *the elementary and thin equivalence have a topological nature, hence they are independent from the choice of the gauge group*, instead *the holonomic equivalence has geometrical character, thus it can depend on the choice of the gauge group, and in fact, as will be proved later, it does.* This is the reason why in the symbol chosen to denote the hoop group it appears G.

A second trivial observation is that the elementary equivalence implies both the thin and the holonomic equivalence, the problem is to understand if, or under what conditions, the converse is true.

A key step in the analysis of this problem is given by this theorem, see [1] or [77] for a proof.

Theorem 4.4.4 *For every finite family $\{g_1, \ldots, g_n\} \subset G$, there is a finite independent family of loops $\{\beta_1, \ldots, \beta_n\} \subset L_\star(M)$ and a connection A such that*

$$g_i = H_A(\beta_i) \qquad i = 1, \ldots, n.$$

This fact will be called **interpolation property**.

Corollary 4.4.1 *For every homomorphism $H : L_\star(M) \to G$ and every finite family of loops $\{\alpha_1, \ldots, \alpha_n\} \subset L_\star(M)$ there exists a connection A such that*

$$H(\alpha_i) = H_A(\alpha_i) \qquad i = 1, \ldots, n\,.$$

In other words, the action of an algebraic homomorphism from $L_\star(M)$ to G on a finite family of loops is identical to that of a certain holonomy map.

Proof. It is sufficient to factorize every loop of the finite family $\{\alpha_1, \ldots, \alpha_n\}$ as the product of certain independent loops $\{\beta_j\}$. In fact, by defining $g_j := H(\beta_j)$ and applying the last theorem there exists a connection $A \in \mathcal{A}$ such that $H_A(\beta_j) = g_j$ for every j. As a consequence of the factorization property of the holonomy map it follows that $H(\alpha_i) = H_A(\alpha_i)$ $i = 1, \ldots, n$. □

The theorem that describes the relations between the loop groups can be stated as follows.

Theorem 4.4.5 *The following assertions hold:*

1. *if G contains a subgroup isomorphic to $SU(2)$ then $\mathcal{H}_\star(M, G)$ is isomorphic to $L_\star(M)$;*

2. *$L_\star(M)$ and $\mathcal{L}_\star(M)$ are always isomorphic.*

I stress that the theorem contemplates the gauge theories with $G = SU(N)$ (and so the Yang-Mills theories), but not the case $G = U(1)$ (or, generally, G Abelian) and in fact $L_\star(M) \neq \mathcal{H}_\star(M, U(1))$ because the last group is Abelian, while the first and the second aren't.

4.5 The role of the holonomy in gauge theory: Kobayashi's representation theorem

In this chapter is investigated the equivalence between gauge-equivalence classes of connections and conjugation classes of holonomy maps, this equivalence will show the great importance of the holonomies in the modern formulation of gauge theories.

The representation theorem deals with this problem: is it possible to know the connection which induces a holonomy map by knowing only the action of this map on the group of loops? The answer is negative for a single connection, what is true is that there is a one-to-one correspondence between holonomies and the set of connections modulo some special kinds of gauge transformations. More important, the conjugation classes of holonomy maps are in one to one correspondence with the gauge equivalence classes of connections, as proved for the first time by Kobayashi [56] in 1954 (see also [70] for a modern review of Kobayashi's work).

To make this assertions precise let us introduce the normal subgroup $\mathcal{G}_\star$ of $\mathcal{G}$ given by the gauge transformations on P which act as the identity on the fiber over the point $\star \in M$.

The first result which leads to Kobayashi's reconstruction theorem is the following.

Lemma 4.5.1 *Let $P \equiv P(M,G)$ and $P' \equiv P'(M,G)$ be two principal bundles with the same base and gauge group but not necessary equal total spaces. Let also A and A' be two connections of P and P', respectively. Finally fix a point $\star \in M$ and two arbitrary points $p_0 \in P_\star$, $p'_0 \in P'_\star$.*

Then $H_A = H_{A'}$ if and only if there exists a G-equivariant isomorphism $\varphi : P \to P'$ which induces the identity on M and such that:

$$\begin{cases} \varphi^* A' = A \\ \varphi(p_0) = p'_0 \, . \end{cases}$$

Proof.

$\Leftarrow$: suppose that there exists an isomorphism φ with the required properties, then it would map A-horizontal paths in P into A'-horizontal paths in P' making the following diagram:

$$\begin{array}{ccc} P'_\star & \xrightarrow{\wp_{\alpha,A'}} & P'_\star \\ {\scriptstyle\varphi}\uparrow & & \uparrow{\scriptstyle\varphi} \\ P_\star & \xrightarrow[\wp_{\alpha,A}]{} & P_\star \end{array}$$

commutative, i.e. $\varphi(\wp_{\alpha,A}(p_0)) = \wp_{\alpha,A'}(\varphi(p_0))$ for every fixed point p_0 in $P_\star$.

Moreover, by definition of parallel transport and thanks to the equivariance of φ one would have $\varphi(\wp_{\alpha,A}(p_0)) = \varphi(p_0.H_A(\alpha)^{-1}) = \varphi(p_0).H_A(\alpha)^{-1} = p'_0.H_A(\alpha)^{-1}$.

Furthermore $\wp_{\alpha,A'}(\varphi(p_0)) = \wp_{\alpha,A'}(p'_0) = p'_0.H_{A'}(\alpha)^{-1}$.

The commutativity of the diagram implies that $p'_0.H_A(\alpha)^{-1} = p'_0.H_{A'}(\alpha)^{-1}$, and so, because of the freedom of the action, $H_{A'}(\alpha) = H_A(\alpha)$, for every $\alpha \in L_\star(M)$, hence $H_A = H_{A'}$.

$\Rightarrow$: suppose now that $H_A = H_{A'}$, then one has to show that it exists a G-equivariant isomorphism with the properties listed in the thesis.

First of all observe that, for every $g \in G$, the map

$$\begin{array}{rccc} \varphi_\star : & P_\star & \longrightarrow & P'_\star \\ & p_0 . g & \mapsto & \varphi_\star(p_0.g) := p'_0 . g \end{array}$$

is a G-equivariant diffeomorphism, thanks to the fact that the action of G is free and transitive on the fibers.

Remember also that, fixed an arbitrary path γ starting in $\star$ and ending in any other fixed point $x \in M$, the parallel transport along γ associated to every connection is also a G-equivariant diffeomorphism, hence the map $\varphi_x : P_x \to P'_x$, $\varphi_x := \wp_{\gamma,A'} \circ \varphi_\star \circ {\wp_{\gamma,A}}^{-1}$ is the composition of G-equivariant diffeomorphisms and so it is a G-equivariant diffeomorphism itself.

The construction is resumed by the following commutative diagram

$$\begin{array}{ccc} P_x & \xrightarrow{\varphi_x} & P'_x \\ {\scriptstyle \wp_{\gamma,A}} \uparrow & & \uparrow {\scriptstyle \wp_{\gamma,A'}} \\ P_\star & \xrightarrow[\varphi_\star]{} & P'_\star . \end{array}$$

φ_x is independent from the choice of the path γ, in fact the contribution of γ is contained only in the parallel transports $\wp_{\gamma,A'}$ and ${\wp_{\gamma,A}}^{-1}$ but these produces two elements in G which are one the inverse of the other, because we are working in the hypothesis that $H_A = H_{A'}$.

It follows that the map $\varphi : P \to P$ defined by $\varphi|_{P_x} := \varphi_x$, $\forall x \in M$, is a G-equivariant isomorphism $\varphi : P \to P$ which projects on the identity of M through the relation 1.1, this follows immediately from the fact that every φ_x has domain and range in a fiber over the same point x (although in different total spaces).

By construction $\varphi \circ \wp_{\alpha,A} = \wp_{\alpha,A'} \circ \varphi$ and, thanks to the properties of the parallel transport, $\varphi^* A' = A$.

Hence φ has all the properties required in the thesis of the theorem. □

If, in particular, one chooses $P' = P$, then one has to fix only the point $p_0 \in P_\star$ and so φ acts as the identity on the fiber $P_\star$. Hence the lemma just proven asserts that $H_A = H_{A'}$ if and only if A and A' are related each other by a gauge transformation $\varphi \in \mathcal{G}_\star$, hence:

$$\mathcal{A}/\mathcal{G}_\star \simeq Hom_P(L_\star(M), G),$$

where $Hom_P(L_\star(M), G)$ denotes the subset of $Hom(L_\star(M), G)$ given by the holonomy maps, thus the orbits of $\mathcal{A}$ w.r.t. $\mathcal{G}_\star$ are faithfully represented by their corresponding holonomy maps.

As an immediate corollary one has the following result.

Corollary 4.5.1 *A choice of a trivialization in a given point $\star \in M$ enables the identification of a connection with its holonomy map along a loop based in $\star$; analogously, a trivialization in the starting and the ending point of a path in M enables the identification of a connection with its parallel transport along the path.*

The previous result can be refined as stated below.

Lemma 4.5.2 *Let the hypothesis of the previous lemma be satisfied, then it exists a G-equivariant isomorphism $\varphi : P \to P'$ inducing the identity on M and satisfying $\varphi^* A' = A$ if and only if both the following conditions are satisfied:*

1. *there exists a G-equivariant map $\varphi_\star : P_\star \to P'_\star$;*
2. *$H_{A'}(\alpha) = Ad_{g^{-1}} H_A(\alpha)$, $\forall \alpha \in L_\star(M)$, where g is the only element of G such that: $p'_0 = \varphi(p_0).g$.*

Proof.
$\Rightarrow$: first of all observe that, by definition of holonomy, $\wp_{\alpha,A}(p_0) = p_0.H_A(\alpha)^{-1}$, hence $\wp_{\alpha,A}(p_0).H_A(\alpha) = p_0.H_A(\alpha)^{-1} H_A(\alpha) = p_0$. Suppose now that there exists a map φ with the required properties, then $\varphi_\star$ can be defined simply as the restriction of it to the fiber $P_\star$ and

$$p'_0 := \varphi(p_0).g = \varphi(\wp_{\alpha,A}(p_0).H_A(\alpha)).g = \varphi(\wp_{\alpha,A}(p_0)).H_A(\alpha)g$$

but, as in the proof of the previous lemma, one finds that $\varphi(\wp_{\alpha,A}(p_0)) = \wp_{\alpha,A'}(\varphi(p_0))$, hence

$$\begin{aligned} p'_0 &= \wp_{\alpha,A'}(\varphi(p_0)).H_A(\alpha)g = \wp_{\alpha,A'}(p'_0.g^{-1}).H_A(\alpha)g = \\ &= \wp_{\alpha,A'}(p'_0).g^{-1} H_A(\alpha)g = p'_0.H_{A'}(\alpha)^{-1} g^{-1} H_A(\alpha)g\,. \end{aligned}$$

Writing $p'_0 = p'_0.e$ one has: $p'_0.e = p'_0.H_{A'}(\alpha)^{-1} g^{-1} H_A(\alpha)g$ it follows that $e = H_{A'}(\alpha)^{-1} g^{-1} H_A(\alpha)g$, but then $H_{A'}(\alpha) = g^{-1} H_A(\alpha)g$ for every loop.

$\Leftarrow$: suppose that there exists a G-equivariant map $\varphi_\star : P_\star \to P'_\star$ and that $H_{A'}(\alpha) = g^{-1} H_A(\alpha)g$ for every loop, then, for any path γ such that $\gamma(0) = \star$ and $\gamma(1) = x$, define $\varphi_x : P_x \to P'_x$, $\varphi_x = \wp_{\gamma,A'} \circ \varphi_\star \circ \wp_{\gamma,A}{}^{-1}$. Following the same arguments of the proof of the previous lemma one easily obtains the thesis. □

As a particular case of the last lemma one can take $P \equiv P'$, then φ becomes a gauge transformation Φ and the next, important, theorem follows immediately.

Theorem 4.5.1 (Representation theorem) *Let A and A' be two connections on a principal fiber bundle $P(M,G)$, then it exists a gauge transformation Φ of P such that*

$$A = \Phi^* A' \qquad \textit{(i.e. } A \textit{ and } A' \textit{ are gauge-equivalent connections)}$$

if and only if there exists $g \in G$ such that $H_{A'}(\alpha) = Ad_{g^{-1}} H_A(\alpha)$, $\forall\, \alpha \in L_\star(M)$.

The representation theorem can be synthetically symbolized as:

$$\mathcal{A}/\mathcal{G} \simeq Hom_P(L_\star(M), G)/Ad_G \, .$$

As discussed in chapter 1., $\mathcal{A}/\mathcal{G}$ is the configuration space of the gauge theories, thus the representation theorem shows the remarkable fact that

the physically distinct configurations of the classical gauge theories are in one-to-one correspondence with the conjugation classes of holonomy maps.

4.6 The Wilson functions

A step of paramount importance for the development of the loop quantization of gauge theories is the recognition of a separating set of gauge-invariant functions of connections.

If the gauge group of a gauge theory is $U(N)$ or $SU(N)$, this functions happen to be the Wilson functions, also known in lattice gauge theory as **Wilson's loop**. The reason of this name relies in the fact that every Wilson function T_α is labelled by a loop α in M and it is defined to be the complex valued function on $\mathcal{A}/\mathcal{G}$ which maps a gauge-equivalence class of connections $[A]$ into the normalized trace of the holonomy $H_A(\alpha)$, where A is any representative of $[A]$, i.e.

$$\begin{array}{rccl} T_\alpha : & \mathcal{A}/\mathcal{G} & \longrightarrow & \mathbb{C} \\ & [A] & \mapsto & T_\alpha([A]) := \frac{1}{N} Tr(H_A(\alpha)) \end{array}$$

where Tr means the trace operator taken in the fundamental representation of the gauge group G.

The definition of T_α is well posed because, if A and A' are two gauge-equivalent connections, then the representation theorem assures that there exist an element $g \in G$ such that $H_{A'}(\alpha) = g^{-1} H_A(\alpha) g$, but, thanks to the cyclic property of the trace, one gets $Tr(H_{A'}(\alpha)) = Tr(g^{-1} H_A(\alpha) g) =$

$Tr(gg^{-1}H_A(\alpha)) = Tr(H_A(\alpha))$. This property is summarized by saying that **the Wilson functions are gauge-invariant**.

Since the Wilson functions are labelled by loops, it is worth remembering that we are dealing with piecewise analytic loops. The regularity of the loops has many remarkable consequences in the development of the loop quantization. The analysis of the piecewise smooth regularity is still under investigation.

Notice that the definition is well posed for loops α belonging to $L_\star(M) \equiv \mathcal{L}_\star(M)$ and also for loops α belonging to $\mathcal{H}_\star(M, G)$, in fact:

- suppose $\alpha \sim_{el} \beta$ then there exists an immediately retraced loop $\gamma = \prod_i \gamma_i {\gamma_i}^{-1}$ such that $\alpha = \beta\gamma$, applying H_A and using his factorization property one has $H_A(\alpha) = H_A(\beta) \prod_i H_A(\gamma_i) H_A(\gamma_i)^{-1} = H_A(\beta)$, hence $T_\alpha = T_\beta$;

- suppose now $\alpha \sim_{hol} \beta$, then, by definition, $H_A(\alpha) = H_A(\beta)$ for every $A \in \mathcal{A}$, thus $T_\alpha = T_\beta$.

Observe also that, since the holonomy maps are unitary representations of the loop group, the Wilson functions induce, by duality, the maps

$$\begin{array}{rccl} T^{[A]}: & L_\star(M) & \longrightarrow & \mathbb{C} \\ & \alpha & \mapsto & T^{[A]}(\alpha) := T_\alpha([A]) \end{array}$$

which are the normalized characters of the unitary representations of the loop group given by the holonomy maps.

The Wilson functions belong to $\mathcal{C}_b(\mathcal{A}/\mathcal{G})$, the continuity follows obviously from the continuity of the trace and the boundness follows from the fact that Tr is invariant under the choice of the base w.r.t. the matrix $H_A(\alpha)$ is represented, thus one can choose the base in which $H_A(\alpha)$ is diagonal, in this case $Tr(H_A(\alpha))$ is the sum of the eigenvalues of the matrix, which are all bounded in absolute value by 1 thanks to the fact that, for any $n \times n$ unitary matrix U, one has $U^\dagger U = I$, i.e. $\sum_{k=1}^n \overline{u_{ik}} u_{jk} = \delta_{ij}$, hence $|Tr(H_A(\alpha))| \leq N$ and

$$\|T_\alpha(A)\|_\infty := \sup_{[A]\in\mathcal{A}/\mathcal{G}} |T_\alpha([A])| \leq 1$$

thanks to the normalization factor.

A very important fact about Wilson functions is that they are subjected to the so-called **Mandelstam identities** of the first and the second kind.

The Mandelstam identities of the first kind hold for every gauge group and they are a simple consequence of the cyclic property of the trace:

$$T_{\alpha\beta} = T_{\beta\alpha}$$

for every couple of loops α and β.

The proof is very easy: by using the factorization property of the holonomy maps and the already mentioned cyclic property of the trace one has, for every $[A] \in \mathcal{A}/\mathcal{G}$, one has:

$$\begin{aligned} T_{\alpha\beta}([A]) &= \tfrac{1}{N}Tr(H_A(\alpha\beta)) = \tfrac{1}{N}Tr(H_A(\alpha)H_A(\beta)) \\ &= \tfrac{1}{N}Tr(H_A(\alpha)H_A(\beta)H_A(\beta)^{-1}H_A(\beta)) \\ &= \tfrac{1}{N}Tr(H_A(\beta)H_A(\alpha)H_A(\beta)H_A(\beta)^{-1}) \\ &= \tfrac{1}{N}Tr(H_A(\beta)H_A(\alpha)) = \tfrac{1}{N}Tr(H_A(\beta\alpha)) \\ &= T_{\beta\alpha}([A])\,. \end{aligned}$$

An immediate consequence of the Mandelstam identities of the first kind is that **the Wilson functions are not independent**, in fact, although the loops $\gamma := \alpha\beta$ and $\eta := \beta\alpha$ are different, the Wilson functions they induce are the same.

To discuss the Mandelstam identities of the second kind one has to distinguish between the monodimensional case, when $G = U(1)$, and the other situations.

When $G = U(1)$ the Mandelstam identities of the second kind simply reflect the property that the trace reduces to the identity operator, thus the factorization property of the holonomy maps extends to the Wilson functions:

$$T_{\alpha\beta} = T_\alpha T_\beta \qquad \text{if } G \equiv U(1)$$

for every loop α and β.

When $N > 1$, the Mandelstam identities of the second kind instead follows from combinatorial arguments strongly depending on the group structure, these arguments are quite technical and not very useful for the later purposes, the only important thing to mention here is that the Mandelstam identities of the second kind relative to any subgroup of $SL(N, \mathbb{C})$ allow to write down the product of the traces of N special matrices as a linear combination of the traces of $N-1$ special matrices, see [42] or [43] for the proof.

In particular this result implies that *the product of the traces of a finite number of* 2×2 *special matrices can be written as a linear combination of the traces of these matrices.*

The immediate consequence is that, *when G is a subgroup of $SL(2, \mathbb{C})$, the algebra generated by the Wilson functions agrees with their linear span.*

As an example the Mandelstam identities of the second kind for $SU(2)$ are:

$$T_\alpha T_\beta = \frac{1}{2}(T_{\alpha\beta} + T_{\alpha\beta^{-1}}) \qquad \text{if } G \equiv SU(2)$$

for every loop α and β, as one can easily verify by direct computation using the Cayley-Klein parameterization of a generic $SU(2)$-matrix, i.e.

$$\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$$

where a and b are complex numbers such that $|a|^2 + |b|^2 = 1$.

This parameterization also shows that, **when $G = SU(2)$, the Wilson functions are real-valued**, in fact the normalized trace of the generic $SU(2)$-matrix above is precisely $\Re(a)$.

To conclude this section dedicated to the Wilson functions it is worth remembering that these ones has been introduced in the mathematical-physics literature in 1974 by Kenneth Wilson in a pioneering article [96] in which he used the traces of the holonomy to construct a manifestly gauge-invariant action of a hyper-cubic lattice gauge theory in the Minkowski space-time exhibiting the confinement of a static couple of quark and anti-quark.

The *confinement of quarks* is the name used by physicists to describe the phenomenon that the quarks never show themselves in a free state, but they are always observed mixed in the adrons, i.e. composites of quarks with null color charge.

Wilson studied a couple of quark and antiquark created at the instant $t = 0$ at distance R and annihilating at the instant $t = \tau$.

The situation is described, on a hypercubic lattice in Minkowski space-time, by the so-called *Wilson action*:

$$S := \frac{1}{N{g_0}^2} \sum_{\alpha} S_\alpha$$

where α is a rectangle in the lattice of vertexes i, j, k, l and where:

$$S_\alpha := -tr(U_{ij} U_{jk} U_{kl} U_{li})$$

being g_0 the coupling constant of the interaction quark-antiquark and U_{ij} the parallel transport matrix (w.r.t. a fixed gauge potential A^a_μ) relative to the segment starting in i and ending in j (analogous for the other matrices).

The *vacuum expectation value* of the Wilson functions in this theory is given by:

$$< T_\alpha > := \frac{\int \frac{1}{N} tr(\prod_\alpha U_{kl})\, e^{-S} \prod_\alpha d\mu_H}{\int e^{-S} \prod_\alpha d\mu_H}$$

where:

- $\prod_\alpha U_{kl}$ symbolizes the ordered product of the matrices associated to the segments composing the loop α in the lattice (*the discrete version of the path-ordered exponential*);

- $\prod_\alpha d\mu_H$ represents the product Haar measure (a Haar measure for every segment which appear in the loop α).

The computation of the binding energy $E(R)$ of the static couple of quark and antiquark in the strong-coupling approximation ($g_0 \to \infty$) and for long times ($\tau \to \infty$) within this model is the following:

$$E(R) \sim Rlog({g_0}^2).$$

This formula shows that the confining potential energy of the couple grows linearly with the distance R between the particles thus prohibiting their macroscopic disjoining.

To get more information about the development of this important topic the interest reader is referred to [61] and [99].

4.7 The overcompleteness of the Wilson functions

In this section it will be proved the overcompleteness of the Wilson functions by using the abstract theory of topological groups and their representations, this proof strongly depends on the fact that the gauge group is chosen to be a unitary (or special unitary) matrix group.

In literature one speaks of overcompleteness to describe the fact that the Wilson functions can separate the classical degrees of freedom of the gauge theories, i.e. the elements of $\mathcal{A}/\mathcal{G}$, but they are not independent.

Since the Wilson functions induce the normalized characters $T^{[A]}$ of the representations of the loops groups given by the holonomy maps, if the loop group was compact then the completeness of the Wilson functions would be immediate since, for compact groups, the characters are in bijection with the equivalence classes of representations.

However the loop group is a rather complicated non-compact group, hence the proof of the completeness of Wilson functions is a non-trivial result.

Let G be an arbitrary group and let $U : \mathsf{G} \to \mathcal{U}(\mathcal{H})$ denote a unitary representation of G into the group $\mathcal{U}(\mathcal{H})$ of the unitary operators on a Hilbert space $\mathcal{H}$ of finite dimension N.

To every such representation U one can associate its **normalized character**, i.e. the map:

$$\begin{array}{rccl} \tau_U : & \mathsf{G} & \longrightarrow & \mathbb{C} \\ & g & \mapsto & \tau_U(g) := \frac{1}{N} Tr(U(g)). \end{array}$$

Two representations $U, \tilde{U}$ of G are said to be **equivalent** if there exists an *invertible* intertwining operator A on $\mathcal{H}$ between them, i.e. an invertible operator A such that $\tilde{U}(g) = AU(g)A^{-1}$, for every $g \in \mathsf{G}$.

From the cyclic property of the trace it follows that two equivalent representations $U, \tilde{U}$ have the same normalized character: $\tau_U = \tau_{\tilde{U}}$.

An equivalence class of unitary representations of G will be indicated by λ, one of its representative by U^λ and the normalized character of one of its representatives by τ_λ.

It is an important fact that, for unitary representations, equivalence classes and unitary equivalence classes agree, as stated in the following theorem.

Theorem 4.7.1 *Two unitary representations U and $\tilde{U}$ of G belong to the same equivalence class if and only if they belong to the same unitary equivalence class.*

Proof.

$\Leftarrow$: U and $\tilde{U}$ belong to the same unitary equivalence class if there exists an *unitary* intertwining operator A such that $\tilde{U}(g) = AU(g)A^{-1}$, for every $g \in \mathsf{G}$. The unitary operators are invertible, hence the implication $\Leftarrow$ is obvious.

$\Rightarrow$: now assume U and $\tilde{U}$ to be equivalent representations of G with intertwining operator given by the *invertible* operator A.

By taking the adjoint of both sides of the intertwining relation one has: $\tilde{U}(g)^\dagger = (A^{-1})^\dagger U(g)^\dagger A^\dagger = (A^\dagger)^{-1} U(g)^\dagger A^\dagger$.

Since $U(g)$ and $\tilde{U}(g)$ are unitary operators they satisfy the identities $U(g)^\dagger = U(g)^{-1}$ and $\tilde{U}(g)^\dagger = \tilde{U}(g)^{-1}$ thus the previous relation can be written as: $\tilde{U}(g)^{-1} = (A^\dagger)^{-1} U(g)^{-1} A^\dagger$.

By taking the inverse of both sides one obtains $\tilde{U}(g) = (A^\dagger)^{-1} U(g) A^\dagger$, which, conjugated by $AA^\dagger$, transforms into $AA^\dagger \tilde{U}(g)(AA^\dagger)^{-1} = AU(g)A^{-1} = \tilde{U}(g)$ so that $\tilde{U}(g)$ commutes with $AA^\dagger$.

Thanks to the spectral theorem this also implies that $|A| := \sqrt{AA^\dagger}$ commutes with $\tilde{U}(g)$, hence the polar decomposition of the invertible operator A can be written as: $A = |A|B$, where B is a unitary operator.

By substituting this expression of A in the intertwining relation one gets $\tilde{U}(g) = |A|BU(g)(|A|B)^{-1}$, i.e. $\tilde{U}(g) = |A|BU(g)B^{-1}|A|^{-1}$, that is

$|A|^{-1}\tilde{U}(g)|A| = BU(g)B^{-1}$. But $[\tilde{U}(g), |A|] = 0$ hence $|A|^{-1}\tilde{U}(g)|A| = |A|^{-1}|A|\tilde{U}(g) = \tilde{U}(g)$.

Since B is unitary, the relation $\tilde{U}(g) = BU(g)B^{-1}$ proves that $U(g)$ and $\tilde{U}(g)$ are unitary equivalent. □

Now it is worth introducing the concept of the compact group associated to any topological group: let $\mathfrak{G}$ be a topological group and let $\mathcal{B}(\mathfrak{G})$ be the space of the bounded complex-valued functions on $\mathfrak{G}$.

The left translation by the element $g \in \mathfrak{G}$ defines an action of $\mathfrak{G}$ on $\mathcal{B}(\mathfrak{G})$ given by:

$$\begin{array}{rcl} \mathfrak{G} \times \mathcal{B}(\mathfrak{G}) & \longrightarrow & \mathcal{B}(\mathfrak{G}) \\ (g, f) & \mapsto & L_g f \end{array}$$

where $L_g f : \mathfrak{G} \to \mathbb{C}$, $L_g f(h) := f(g^{-1}h)$, for every $h \in \mathfrak{G}$.

This action is also well defined, by restriction, on $\mathcal{C}_b(\mathfrak{G})$, the space of bounded continuous complex-valued functions on $\mathfrak{G}$.

$\mathcal{B}(\mathfrak{G})$ and $\mathcal{C}_b(\mathfrak{G})$ are Abelian C^*-algebras w.r.t. the $\| \|_\infty$-norm and the correspondence $f \to L_g f$ is an isometry for both of them.

Def. 4.7.1 *The* **left orbit** *of a function $f \in \mathcal{B}(\mathfrak{G})$ is the closure of the set $\{L_g f \mid g \in \mathfrak{G}\}$ w.r.t. the $\| \|_\infty$-norm. It is denoted by X_f.*

If X_f is a compact set then f is said to be **almost periodic**. *The set of all almost periodic functions on $\mathfrak{G}$ is denoted by $A(\mathfrak{G})$.*

Standard examples of almost periodic functions are the characters of the continuous representations of a compact group.

It is obvious that the left translations carry $A(\mathfrak{G})$ into itself and acts isometrically on every orbit X_f, hence the closure of the group of the left translations in the topology of the punctual convergence, denoted by $\chi(\mathfrak{G})$, is a closed subgroup of $\prod_{f \in A(\mathfrak{G})} Iso(X_f)$, where $Iso(X_f)$ is the group of the isometries of X_f, which is a compact Hausdorff group in the topology of punctual convergence by a standard theorem of topology.

It follows that also $\chi(\mathfrak{G})$ is a compact Hausdorff group (w.r.t. the induced topology) and it is called **the compact group associated to the topological group $\mathfrak{G}$**.

It can be proved that the map

$$\begin{array}{rrcl} \chi : & \mathfrak{G} & \longrightarrow & \chi(\mathfrak{G}) \\ & g & \mapsto & \chi(g) \end{array}$$

where $\chi(g) := \{L_g|_{X_f}, f \in A(\mathfrak{G})\}$, is a continuous homomorphism with dense range.

Furthermore $\chi(\mathfrak{G})$ has the following universal property ([54]):

Theorem 4.7.2 *If $\Phi : \mathfrak{G} \to K$ is a continuous homomorphism from the topological group $\mathfrak{G}$ into the compact group K, then it always exists a continuous homomorphism $\varphi : \chi(\mathfrak{G}) \to K$ such that the following diagram*

$$\begin{array}{ccc} \chi(\mathfrak{G}) & \xrightarrow{\varphi} & K \\ {\scriptstyle\chi}\uparrow & & \uparrow{\scriptstyle\Phi} \\ \mathfrak{G} & = & \mathfrak{G} \end{array}$$

commutes, i.e. $\varphi \circ \chi = \Phi$.

This results allows to extend a well known theorem involving unitary representations of compact groups to unitary representations of every topological groups.

Theorem 4.7.3 *Let $\mathfrak{G}$ be an arbitrary topological group. Let also U and $\tilde{U}$ be two continuous unitary representations of $\mathfrak{G}$ with normalized characters τ_U and $\tau_{\tilde{U}}$. If $\tau_U = \tau_{\tilde{U}}$, then U and $\tilde{U}$ belong to the same unitary equivalence class.*

Proof. Thanks to the universal property of the compact group associated to $\mathfrak{G}$ one can extend every continuous unitary representation $U : \mathfrak{G} \to U(N)$ of $\mathfrak{G}$ to the unique continuous unitary representation $V : \chi(\mathfrak{G}) \to U(N)$ of $\chi(\mathfrak{G})$ that makes the following diagram

$$\begin{array}{ccc} \chi(\mathfrak{G}) & \xrightarrow{V} & U(N) \\ {\scriptstyle\chi}\uparrow & & \uparrow{\scriptstyle U} \\ \mathfrak{G} & = & \mathfrak{G} \end{array}$$

commute, i.e. $U = V \circ \chi$. The same considerations holds for $\tilde{U}$ with analogous notations.

Let T_V be the normalized character of V, then $(\frac{1}{N}Tr) \circ U = (\frac{1}{N}Tr) \circ V \circ \chi$, i.e. $\tau_U = \mathsf{T}_V \circ \chi$ and the same thing holds for the normalized character $\mathsf{T}_{\tilde{V}}$ of $\tilde{V}$.

Thanks to the fact that $\mathfrak{G}$ is dense in $\chi(\mathfrak{G})$, $\tau_U = \tau_{\tilde{U}}$ implies $\mathsf{T}_V = \mathsf{T}_{\tilde{V}}$, but these are the normalized character of continuous representations of compact groups, hence V and $\tilde{V}$ are equivalent for a well known result (see for instance [28] or [88]).

Thanks to the commutativity of the last diagram, this implies that also U and $\tilde{U}$ are equivalent, hence, by theorem 4.7.1, it follows that V and $\tilde{V}$ belong to the same unitary equivalence class. □

The previous theorem can be generalized to *every* group G.

Corollary 4.7.1 *Let* G *be an arbitrary group. Let also* $U : \mathsf{G} \to U(N)$ *and* $\tilde{U} : \mathsf{G} \to U(N)$ *be two unitary representations with normalized characters* τ_U *and* $\tau_{\tilde{U}}$*. If* $\tau_U = \tau_{\tilde{U}}$*, then* U *and* $\tilde{U}$ *belong to the same unitary equivalence class.*

Proof. Endow G with the topology induced by all the unitary representations $\mathsf{G} \to U(N)$ (i.e. the smallest topology which makes them continuous). Then G becomes a topological group to which the previous theorem applies. □

The abstract theory developed until now can be specialized to the case in which the group is one of the loop groups, its unitary representations are the holonomy maps and their normalized characters are the functions $T^{[A]}$ induced by duality from the Wilson functions.

With these choices one immediately obtains that every function $T^{[A]}$ univocally characterizes a conjugation class of holonomy maps, i.e. a point in $Hom_P(L_\star(M), G)/Ad_G$, i.e. a gauge equivalence class of connections, hence the correspondence $[A] \mapsto T^{[A]}$ is bijective.

Note, however, that the set $\{T_\alpha \mid \alpha \in L_\star(M)\}$ is bigger than the set $\{T^{[A]} \mid [A] \in \mathcal{A}/\mathcal{G}\}$, because the Mandelstam identities implies that to different loops can correspond identical Wilson functions, hence the first set contains redundant copies of the same Wilson functions.

An immediate mathematical consequence of the bijection established above is that **for unitary gauge theories, the Wilson functions are separating on** $\mathcal{A}/\mathcal{G}$, i.e. $[A_1], [A_2] \in \mathcal{A}/\mathcal{G}$, $[A_1] \neq [A_2]$ implies that there exists at least a Wilson function T_α such that $T_\alpha([A_1]) \neq T_\alpha([A_2])$.

As remarked at the begin of this section, this behavior is often summarized by saying that **the Wilson functions form an overcomplete set of gauge invariant functions**.

4.8 The holonomy C^*-algebra $Hol(M, G)$ and its spectrum $\overline{\mathcal{A}/\mathcal{G}}$

By taking all the linear combinations of finite products of Wilson functions one gets an unital Abelian algebra w.r.t. punctual multiplication and with unit element given by the Wilson function associated to the unit loop $\star \in L_\star(M)$, this is the constant function $T_\star([A]) \equiv 1$, $\forall [A] \in \mathcal{A}/\mathcal{G}$.

This is also a $*$-algebra, indicated by $hol(M, G)$, w.r.t. complex conjugation and it is easy to see that

$$T_\alpha{}^* = T_{\alpha^{-1}}\,.$$

The completion of this $*$-algebra w.r.t. the topology induced by the $\| \|_\infty$ norm gives rise to a unital Abelian C^*-algebra called **holonomy C^*-algebra** and denoted by $Hol(M, G)$ because it depends both on M and G, but not on the principal bundle $P(M, G)$, as will be proved later. More rigorously this is the *analytic holonomy C^*-algebra* because the loops taken into account are assumed to be piecewise analytic.

Now it's useful to remember from chapter 3 that **the states of an Abelian C^*-algebra $\mathfrak{A}$ are in bijection with the probability measures on its spectrum** $\sigma(\mathfrak{A})$ and that, to every positive measure μ on $\sigma(\mathfrak{A})$ (alias to every positive functional φ_μ on $\mathfrak{A}$) one can associate the so-called **GNS representation**, which is given by the correspondence $a \mapsto M_{\hat{a}}$, where $M_{\hat{a}}$ is the multiplication operator on $L^2(\sigma(\mathfrak{A}), \mu)$ defined by $M_{\hat{a}}\psi := \hat{a}\psi$, for every $a \in \mathfrak{A}$ and $\psi \in L^2(\sigma(\mathfrak{A}), \mu)$.

From a physical point of view the GNS construction relates the C^*-algebraic approach to quantum physics to the standard one based on Hilbert spaces.

The abstract results of C^*-algebras theory apply to the holonomy C^*-algebra $Hol(M, G)$, whose compact Hausdorff spectrum $\sigma(Hol(M, G))$ is usually written $\overline{\mathcal{A}/\mathcal{G}}$ for reasons that will be cleared in the next section.

Denoting by $\bar{A}$ the elements of $\overline{\mathcal{A}/\mathcal{G}}$, the Gelfand isomorphism specialized to the holonomy C^*-algebra can be written as:

$$\begin{array}{rlll} \hat{}: & Hol(M,G) & \longrightarrow & \mathcal{C}(\overline{\mathcal{A}/\mathcal{G}}) \\ & f & \mapsto & \hat{f}, \quad \hat{f}(\bar{A}) := \bar{A}(f). \end{array}$$

The isometric isomorphism $Hol(M, G) \simeq \mathcal{C}(\overline{\mathcal{A}/\mathcal{G}})$ will be used many times in the sequel.

In particular note that the Gelfand isomorphism implies that every function $f \in Hol(M, G)$, which is defined on $\mathcal{A}/\mathcal{G}$, can be extended in a unique way to the *continuous* function $\hat{f}$ on $\overline{\mathcal{A}/\mathcal{G}}$.

4.9 The dense injection of $\mathcal{A}/\mathcal{G}$ in $\overline{\mathcal{A}/\mathcal{G}}$ and its algebraic characterization

The results presented in this section has been first discussed by Ashtekar and Lewandowski in [7], then Baumgärtel encoded their arguments in a more abstract and rigorous setting (see [26]).

We will follow Baumgärtel's construction, but working directly on the concrete objects of the gauge theories instead of treating the problem abstractly.

Theorem 4.9.1 *The map*

$$\begin{array}{rcl} T: \ \mathcal{A}/\mathcal{G} \simeq Hom_P(L_\star(M), G)/Ad_G & \longrightarrow & \sigma(Hol(M,G)) \\ {[A]} & \mapsto & T([A]) := T_{[A]} \end{array}$$

where $T_{[A]}$ is the linear functional on $Hol(M,G)$ defined by

$$\begin{array}{rcl} T_{[A]}: \ Hol(M,G) & \longrightarrow & \mathbb{C} \\ f & \mapsto & T_{[A]}(f) := f([A]) \end{array}$$

is a dense injection of $\mathcal{A}/\mathcal{G}$ into $\sigma(Hol(M,G))$, the spectrum of the holonomy C^*-algebra.

Proof. The map T is well defined, in fact $Hol(M,G)$ is an algebra under pointwise multiplication, thus $T_{[A]}(fh) = (fh)([A]) = f([A])h([A]) = T_{[A]}(f)T_{[A]}(h)$, $f,h \in Hol(M,G)$. Furthermore it is evident that $T_{[A]}$ is continuous since it acts as the evaluation map, hence it is a character of $Hol(M,G)$.

T is injective: if, by absurd, $[A],[A]' \in \mathcal{A}/\mathcal{G}$, $[A] \neq [A]'$, but $T_{[A]} = T_{[A]'}$, then by definition $f([A]) = f([A]')$ $\forall f \in Hol(M,G)$, but this is absurd since the Wilson functions are separating on $\mathcal{A}/\mathcal{G}$ and so there is at least a $T_\alpha \in Hol(M,G)$ such that $T_\alpha([A]) \neq T_\alpha([A]')$.

Finally, T is dense: if, by absurd, $T(\mathcal{A}/\mathcal{G})$ wasn't dense in $\sigma(Hol(M,G))$ then, thanks to Urysohn's Lemma, there would exist a *continuous* function, say F, on $\sigma(Hol(M,G))$ which takes the value 1 on some point outside $T(\mathcal{A}/\mathcal{G})$ and vanish on the entire $T(\mathcal{A}/\mathcal{G})$. But this is absurd since $f \in \mathcal{C}(\sigma(Hol(M,G)))$ and $f + F$ would be two different extensions of same function on $\mathcal{A}/\mathcal{G}$, against the fact that the Gelfand transform is an isomorphism. □

Once established this fact one would obviously like to find a simple characterization of the spectrum of the holonomy C^*-algebra. This turns to be possible, in fact $\sigma(Hol(M,G))$ is precisely the set of *all* algebraic homomorphisms $H : L_\star(M) \to G$, i.e. the set of the generalized connections modulo Ad_G-conjugation, in the terminology introduced in 3.2 .

Theorem 4.9.2 *Fixed any character $\varphi \in \sigma(Hol(M,G))$, there is one and only one Ad_G-class of algebraic homomorphisms $[H]_\varphi \in Hom(L_\star(M), G)/Ad_G$ such that: $\varphi(T_\alpha) = \frac{1}{N} Tr([H]_\varphi(\alpha))$ $\forall \alpha \in L_\star(M)$, i.e. the map*

$$\begin{array}{rcl} \sigma(Hol(M,G)) & \longrightarrow & Hom(L_\star(M), G)/Ad_G \\ \varphi & \mapsto & [H]_\varphi, \end{array}$$

is an injection.

The proof of this theorem is quite technical, the interest reader is referred to [1].

There are no indications that the injection from the spectrum $\sigma(Hol(M,G))$ to $Hom(L_\star(M),G)/Ad_G$ established in the previous theorem is also onto, however, thanks to the interpolation property of the holonomies one can prove that the correspondence in exam is also surjective.

Theorem 4.9.3 *The interpolation condition is a sufficient condition for the surjectivity of the map $\sigma(Hol(M,G)) \hookrightarrow Hom(L_\star(M),G)/Ad_G$, which is actually a bijection.*

Proof. The only thing to prove is that, if the interpolation property holds, then to every element of $Hom(L_\star(M),G)/Ad_G$ there correspond a character on $Hol(M,G)$.

First of all remember that every element of the Wilson algebra $hol(M,G)$ is a finite sum of the form

$$p = p_0 + \sum_{j_1} p_{j_1} T_{\alpha_{j_1}} + \sum_{j_1,j_2} p_{j_1,j_2} T_{\alpha_{j_1}} T\alpha_{j_2} + \dots$$

which is called a 'Wilson polynomial'.

For any given representative H of $[H] \in Hom(L_\star(M),G)/Ad_G$, define a functional φ_H on the set of Wilson functions as:

$$\varphi_H(T_\alpha) := \frac{1}{N} Tr(H(\alpha)) \qquad \forall \alpha \in \mathfrak{G}$$

and extend it to the Wilson algebra $hol(M,G)$ by posing:

$$\varphi_H(p) := p_0 + \sum_{j_1} p_{j_1} \varphi_H(T_{\alpha_{j_1}}) + \sum_{j_1,j_2} p_{j_1,j_2} \varphi_H(T_{\alpha_{j_1}}) \varphi_H(T_{\alpha_{j_2}}) + \dots$$

To verify that this definition is well posed observe that any given Wilson polynomial p depends only on a finite number of elements $\alpha_1, \dots, \alpha_r$ in G, hence, as a consequence of the interpolation property, there exists a connection A (in general depending on p) such that

$$\varphi_H(T_{\alpha_k}) = \varphi_{H_A}(T_{\alpha_k}) = \frac{1}{N} Tr(H_A(\alpha_k)) = T_{\alpha_k}(A) \qquad k = 1, \dots, r.$$

Since the Wilson functions are well defined on equivalence classes of connections (or, equivalently, on unitary classes of holonomy maps), every functional φ_H is well defined.

The next step is to prove that φ_H is multiplicative and bounded.

Fix an arbitrary couple p_1, p_2 of Wilson polynomials and use again the interpolation property to choose a connection A such that $\varphi_H(\alpha_l) = \frac{1}{N}Tr(H_A(\alpha_l)) = T_{\alpha_l}(A)$ for every $\alpha_l \in \mathfrak{G}$ from which p_1 and p_2 depend.

From the multiplicative character of H_A one easily obtains $\varphi_H(p_1 p_2) = p_1(H_A)p_2(H_A) = \varphi_H(p_1)\varphi_U(p_2)$, thus φ_H is multiplicative.

Furthermore $|\varphi_H(p)| = |p(H_A)| \leq \|p\|$, hence φ_H is also bounded, alias continuous, so it can be extended in a unique way to a bounded multiplicative functional φ on the C^*-algebra obtained by the completion of $hol(M,G)$, but then φ belongs to $\sigma(Hol(M,G))$ and the theorem is proved. □

It is worth noting that $Hom(L_\star(M),G)/Ad_G$ is endowed with any particular topology, this one can be induced from that of $\sigma(Hol(M,G))$ only after their set-theoretical identification through the theorem just proved.

The results just proved can be resumed in the following important theorem.

Theorem 4.9.4 (Ashtekar-Lewandowski-Baumgärtel) *Whenever G is $U(N)$ or $SU(N)$, the spectrum of the holonomy C^*-algebra $Hol(M,G)$ can be algebraically characterized as the space of all homomorphism from the group of loops $L_\star(M)$ to G modulo conjugation:*

$$\sigma(Hol(M,G)) \simeq Hom(L_\star(M),G)/Ad_G\,.$$

Because of the fact that $\mathcal{A}/\mathcal{G}$ is densely embedded in $\sigma(Hol(M,G))$, in literature it has been chosen the symbol $\overline{\mathcal{A}/\mathcal{G}}$ to shortly denote the spectrum of the holonomy C^*-algebra, so that:

$$\overline{\mathcal{A}/\mathcal{G}} := \sigma(Hol(M,G)) \simeq Hom(L_\star(M),G)/Ad_G$$

the elements of $\overline{\mathcal{A}/\mathcal{G}}$ will be denoted by $\bar{A}$ and called **generalized connections**.

A generalized connection hence can be thought as a character of the holonomy C^*-algebra or as an algebraic homomorphisms from the group of loops to the gauge group modulo Ad_G-equivalence.

Since every connection on every PFB with base M and structure group G induces a holonomy map which happens to be a homomorphism from $L_\star(M)$ to G, the Ashtekar-Lewandowski-Baumgärtel theorem implies that **to every connection on every principal fiber bundle over M with structural group G corresponds a point in $\overline{\mathcal{A}/\mathcal{G}}$!**

Even though this space seems to be 'too big' to be interesting, it is 'small enough' to be endowed with a natural probability measure and a rich differential structure, these are two of the most important reason which makes

$\overline{\mathcal{A}/\mathcal{G}}$ a natural candidate to the role of quantum configuration space of a gauge theory.

The algebraic characterization of $\overline{\mathcal{A}/\mathcal{G}}$ has this important corollary.

Corollary 4.9.1 *Whenever the gauge group G is $U(N)$ or $SU(N)$, $Hol(M,G)$ doesn't depend on the principal fiber bundle $P(M,G)$ but only on M and G.*

Proof. Since the spectrum $\overline{\mathcal{A}/\mathcal{G}}$ of $Hol(M,G)$ is characterized as the set $Hom(L_\star(M),G)/Ad_G$, it depends only on M and G, because this is the only dependence of an element of the last space.

The Gelfand isomorphism now reads:

$$\begin{array}{rccc} \hat{}: & Hol(M,G) & \longrightarrow & \mathcal{C}(\overline{\mathcal{A}/\mathcal{G}}) \\ & f & \mapsto & \hat{f}, \end{array}$$

$\hat{f}(\bar{A}) := \bar{A}(f)$, thus even $Hol(M,G)$ doesn't depend on $P(M,G)$ but only on M and G, since it is isomorphic to $\mathcal{C}(\overline{\mathcal{A}/\mathcal{G}})$ which obviously depends only on M and G. □

The function f which appear in the previous proof is a Wilson function or a uniform limit of a sequence of Wilson functions. If f happens to be a Wilson functions T_α, for a certain loop $\alpha \in L_\star(M)$, then it is possible to write down the explicit action of its Gelfand transformed $\hat{T}_\alpha \in \mathcal{C}(\overline{\mathcal{A}/\mathcal{G}})$, $\hat{T}_\alpha(\bar{A}) := \bar{A}(T_\alpha)$.

In fact it is sufficient to remember that, as shown in the proof of the theorem 4.9.3, the character $\bar{A} \in \sigma(Hol(M,G))$ is in bijection with a Ad_G-equivalence class $[H] \in Hom(L_\star(M),G)/Ad_G$ such that $\bar{A}(T_\alpha) = \frac{1}{N}Tr(H(\alpha))$, for every fixed representative $H \in [H]$, hence:

$$\hat{T}_\alpha(\bar{A}) = \frac{1}{N}Tr(H(\alpha))\,.$$

In the sequel, when there won't be risk of confusion, I shall omit the hat symbol and denote the functions of $Hol(M,G)$ and their Gelfand transformed − which are elements of $\mathcal{C}(\overline{\mathcal{A}/\mathcal{G}})$ − with the same symbol.

4.10 Projective and inductive limits

Now we examine the projective characterization of $\overline{\mathcal{A}/\mathcal{G}}$. This is a very useful (and elegant) identification of the spectrum of the holonomy C^*-algebra with a projective limit.

This result has clarified the structure of $\overline{\mathcal{A}/\mathcal{G}}$ and, most important, has enabled to construct a non-trivial measure on it, the so-called *uniform measure*, which is indispensable for the procedure of quantization of gauge theories that will be described in the following chapter.

Before developing the projective characterization of $\overline{\mathcal{A}/\mathcal{G}}$ we give the most important definitions and results about projective and inductive limits.

The basic definition about projective limits of topological spaces is the following (for a wider and more complete discussion on projective limits the interest reader is referred to [35]).

Def. 4.10.1 *A* **projective family** *of topological Hausdorff spaces is a triple* $\{\Omega_j, \pi_{ij}, J\}$ *where:*

- Ω_j *is a topological Hausdorff space for every* $j \in J$.
- J *is a* **directed** *set of indexes, i.e. it is endowed with a partial order relationship* $\leq$ *such that*

$$\forall i, j \in J \ \exists k \in J \text{ such that } i \leq k \text{ and } j \leq k.$$

- *if* $i \leq j$ *then the maps* $\pi_{ij} : \Omega_j \to \Omega_i$ *are continuous surjective projections such that:*
    1. $\pi_{jj} = id_{\Omega_j}$ $\forall j \in J$;
    2. *if* $i \leq j \leq k$ *then* $\pi_{ij} \circ \pi_{jk} = \pi_{ik}$ *(***"consistency relation"***).*

An element $\{\omega_j\}_{j\in J}$ *of the cartesian product* $\prod_{j\in J} \Omega_j$ *is called* **wire** *if it satisfies the condition*

$$\pi_{ij}\omega_j = \omega_i \qquad \forall i < j$$

i.e. if every element of the ordered sequence is obtained from one of the previous via projection.

The **projective limit** *of* $\{\Omega_j, \pi_{ij}, J\}$ *is the subset of the cartesian product* $\prod_{j\in J} \Omega_j$ *given by all its wires, this space is indicated by*

$$\Omega \equiv \varprojlim_{j\in J} \Omega_j \, .$$

The maps

$$\begin{array}{rccl} \pi_j : & \Omega & \longrightarrow & \Omega_j \\ & \{\omega_i\}_{i\in J} & \mapsto & \pi_j(\{\omega_i\}_{i\in J}) := \omega_j \end{array}$$

are called the projections of Ω.

The projective limit Ω carries a natural topology, called **initial topology**, which is the smallest topology w.r.t. the projections π_j of Ω are continuous.

A base of this topology is given by the sets $\prod_{j\in J} U_j$, where $U_j \in \Omega_j$ is an open set such that $U_j = \Omega_j$ $\forall j \in J$ but for a finite number of indexes.

In the initial topology *the projections are open maps* and *the projective limit is closed.*

It is easy to proof that if I is a **cofinal** subset of J, i.e. $\forall j \in J$ $\exists i \in I$ such that $j \leq i$, then

$$\varprojlim_{j\in J} \Omega_j = \varprojlim_{i\in I} \Omega_i \, .$$

Furthermore, *if the spaces Ω_j are all compact then the projective limit Ω is a non-empty compact Hausdorff space.*

There is a very important class of functions associated to the projective limit of topological spaces, the class of the **cylindrical functions**.

Def. 4.10.2 *The space $Cyl(\Omega)$ of the cylindrical functions on the projective limit Ω of the family $\{\Omega_j, \pi_{ij}, J\}$ is the quotient of the disjoint union $\coprod_{j\in J} \mathcal{C}(\Omega_j)$ modulo the equivalence relation defined by: $f \in \mathcal{C}(\Omega_j)$, $g \in \mathcal{C}(\Omega_{j'})$, $f \sim g$ if there exists an index j'' such that $\pi_{jj''}(f) = \pi_{j'j''}(g)$.*

Note that, in particular, the cylindrical functions are continuous, by converse it can be easily proved that *a continuous function f on Ω is cylindrical if and only if there exists a function $f_j \in \mathcal{C}(\Omega_j)$ such that $f = f_j \circ \pi_j$*, if this is the case then f is said to be cylindrical w.r.t. the index j and one writes $f \in Cyl_j(\Omega)$. Obviously

$$Cyl(\Omega) = \coprod_{j\in J} Cyl_j(\Omega) \, .$$

The map

$$\begin{array}{rccc} i: & Cyl(\Omega) & \longrightarrow & \mathcal{C}(\Omega) \\ & f_j & \mapsto & i(f_j) := f_j \circ \pi_j \end{array}$$

is an injective homomorphism which embeds $Cyl(\Omega)$ in $\mathcal{C}(\Omega)$.

The final result I want to cite about projective limits is the celebrated A.Weil's theorem (see [94]).

Theorem 4.10.1 *Every compact group is the projective limit of compact Lie groups.*

The dual construction of the projective limit is the inductive limit. For the later purposes it is worth introducing the definition of inductive limit directly on C^*-algebras, the same definition extends to general linear spaces and algebras. Here the reference is [64].

Def. 4.10.3 *An* **inductive family** *of C^*-algebras is a triple $\{\mathfrak{A}_\alpha, i_{\beta\alpha}, A\}$ where $\mathfrak{A}_\alpha$ are C^*-algebras and A is a directed set of indexes such that, for every $\alpha \leq \beta$, there exist continuous injective inclusions $i_{\beta\alpha} : \mathfrak{A}_\alpha \to \mathfrak{A}_\beta$ satisfying:*

1. $i_{\alpha\alpha} = id_{\mathfrak{A}_\alpha}$;
2. $i_{\gamma\beta} \circ i_{\beta\alpha} = i_{\gamma\alpha}$, *whenever* $\alpha \leq \beta \leq \gamma$.

The **inductive limit** *of $\{\mathfrak{A}_\alpha, i_{\beta\alpha}, A\}$ is, set-theoretically, the quotient of the disjoint union $\coprod_{\alpha\in A} \mathfrak{A}_\alpha$ modulo the following equivalence relation: $a \in \mathfrak{A}_\alpha$, $b \in \mathfrak{A}_\beta$, $a \sim b$ if there exists $\gamma \geq \alpha, \beta$ such that $i_{\gamma\alpha}(a) = i_{\gamma\beta}(b)$.*

The symbol used to represent the inductive limit is

$$\mathfrak{A} \equiv \varinjlim_{\alpha\in A} \mathfrak{A}_\alpha\,.$$

The canonical inclusion of $\mathfrak{A}_\alpha$, α fixed in A, in the disjoint union defines, by quotient, the inclusion map in the inductive limit $\mathfrak{A}$, $i_\alpha : \mathfrak{A}_\alpha \hookrightarrow \mathfrak{A}$, which satisfies $i_\beta \circ i_{\beta\alpha} = i_\alpha$ for every $\alpha \leq \beta$.

To endow $\mathfrak{A}$ with an algebraic structure it is necessary to use the following lemma.

Lemma 4.10.1 *Let $\{\mathfrak{A}_\alpha, i_{\beta\alpha}, A\}$ be an inductive family of C^*-algebras with inductive limit $\mathfrak{A}$. Then, fixed n elements $\{a_1, \ldots, a_n\} \subset \mathfrak{A}$, there exist an index β and n elements $\{b_1, \ldots, b_n\} \subset \mathfrak{A}_\beta$ such that*

$$a_i = i_\beta(b_i) \qquad i = 1, \ldots, n.$$

Thanks to the previous lemma one can define the $*$-algebraic structure of $\mathfrak{A}$ using that of the $*$-algebras appearing in the family:

$$\begin{cases} \lambda a := i_\beta(\lambda b) \\ a_1 + a_2 := i_\beta(b_1 + b_2) \\ a_1 a_2 := i_\beta(b_1 b_2) \\ a^* := i_\beta(b^*) \end{cases}$$

where $\lambda \in \mathbb{C}$, $a, a_1, a_2 \in \mathfrak{A}$ and $b, b_1, b_2 \in \mathfrak{A}_\beta$ satisfy $i_\beta(b) = a$, $i_\beta(b_1) = a_1$ and $i_\beta(b_2) = a_2$.

By endowing $\mathfrak{A}$ of the finest locally convex topology which makes the homomorphisms i_α continuous, called **final topology**, $\mathfrak{A}$ becomes a topological $*$-algebra.

It is worth noting that **an inductive family of C^*-algebras always induces a projective family**, in fact if $\{\mathfrak{A}_\alpha, i_{\beta\alpha}, A\}$ is such a family then a projective family is obtained by associating to every $\mathfrak{A}_\alpha$ its spectrum $\sigma(\mathfrak{A}_\alpha)$ and to every inclusion $i_{\beta\alpha}$, $\alpha \le \beta$, the restriction of its transposed map to the spectrum of $\mathfrak{A}_\beta$, $\pi_{\alpha\beta} \equiv {}^t i_{\beta\alpha}|_{\sigma(\mathfrak{A}_\beta)}$, where:

$$\begin{array}{rccc} {}^t i_{\beta\alpha}: & {\mathfrak{A}_\beta}^* & \longrightarrow & {\mathfrak{A}_\alpha}^* \\ & \varphi & \mapsto & {}^t i_{\beta\alpha}(\varphi), \end{array}$$

is defined in the usual way, i.e. $({}^t i_{\beta\alpha}(\varphi))(a) := \varphi(i_{\beta\alpha}(a))$, $a \in \mathfrak{A}_\alpha$.

It is easy to verify that the family $\{\sigma(\mathfrak{A}_\alpha), \pi_{\alpha\beta}, A\}$ is a well defined projective family.

If the $\mathfrak{A}_\alpha$ are also unital and Abelian then the spectra $\sigma(\mathfrak{A}_\alpha)$ are compact Hausdorff spaces, hence the projective limit $\varprojlim_{\alpha\in A}\mathfrak{A}_\alpha$ is a non-void compact Hausdorff space.

The most remarkable fact about this family, which will be used in the next section to obtain the projective characterization of $\overline{\mathcal{A}/\mathcal{G}}$, is expressed by the following theorem (Th. 3.43 of [64]).

Theorem 4.10.2 *Let $\{\mathfrak{A}_\alpha, i_{\beta\alpha}, A\}$ be an inductive family of Abelian C^*-algebras with unit. Then its inductive limit $\mathfrak{A}$ is an Abelian topological algebra with unit (in the final topology) whose spectrum $\sigma(\mathfrak{A})$ is a compact Hausdorff space homeomorphic to the projective limit of $\{\sigma(\mathfrak{A}_\alpha), \pi_{\beta\alpha}, A\}$:*

$$\mathfrak{A} = \varinjlim_{\alpha\in A}\mathfrak{A}_\alpha \quad \Rightarrow \quad \sigma(\mathfrak{A}) \simeq \varprojlim_{\alpha\in A}\sigma(\mathfrak{A}_\alpha)\,.$$

4.11 The projective characterization of $\overline{\mathcal{A}/\mathcal{G}}$

In this section it will be shown how $\overline{\mathcal{A}/\mathcal{G}}$ can be identified with a projective limit.

First of all fix the directed set of indexes to be the set of all graphs Γ in M ordered w.r.t. the natural inclusion and denote it by L. This set is directed because if Γ and Γ' belong to L then also $\Gamma \cup \Gamma'$ belongs to L and $\Gamma \le \Gamma \cup \Gamma'$, $\Gamma' \le \Gamma \cup \Gamma'$.

Now the idea is to use this directed set to construct an inductive family of C^*-algebras whose inductive limit is dense in the holonomy C^*-algebra, then, by using theorem 4.10.2, the desired result will be reached.

To every graph Γ associate the unital Abelian C^*-algebra $A(\Gamma)$ generated by the Wilson functions T_α such that $\alpha^* \subset \Gamma$ for, at least, one representative loop in $[\alpha]_{el} \in L_\star(M)$.

It is obvious that if $f \in A(\Gamma)$ then $f \in A(\Gamma')$ for every $\Gamma' \ge \Gamma$ so that the inclusions $i_{\Gamma'\Gamma}$ are naturally defined by:

$$\begin{array}{rccc} i_{\Gamma'\Gamma}: & A(\Gamma) & \hookrightarrow & A(\Gamma') \\ & f & \mapsto & i_{\Gamma'\Gamma}(f) := f\,. \end{array}$$

This inclusions satisfy the consistency relations: $i_{\Gamma''\Gamma'} \circ i_{\Gamma'\Gamma} = i_{\Gamma''\Gamma}$ for every $\Gamma'' \geq \Gamma' \geq \Gamma$.

Moreover the inclusion $i_\Gamma : A(\Gamma) \hookrightarrow Hol(M, G)$, $i_\Gamma(f) := f$, satisfies $i_\Gamma = i_{\Gamma'} \circ i_{\Gamma'\Gamma}$.

Hence $\{A(\Gamma), i_{\Gamma'\Gamma}, L\}$ is an inductive family of unital Abelian C^*-algebras whose inductive limit is continuously included in $Hol(M, G)$.

By comparing the definition of inductive limit of the $A(\Gamma) \subset \mathcal{C}(\mathcal{A}/\mathcal{G})$ and the definition of the algebra of the cylindrical functions on $\mathcal{A}/\mathcal{G}$ one immediately recognizes that the two algebras agree:

$$\varinjlim_{\Gamma \in L} A(\Gamma) = Cyl(\mathcal{A}/\mathcal{G})\,.$$

Observe now that the polynomial algebra $\mathcal{W}$ generated by the Wilson functions is contained in $Cyl(\mathcal{A}/\mathcal{G})$ hence:

$$\overline{Cyl(\mathcal{A}/\mathcal{G})} = Hol(M, G)\,.$$

If $\sigma(\Gamma)$ denotes the (compact, Hausdorff) spectrum of $A(\Gamma)$, then the theorem 4.10.2 implies that

$$\varprojlim_{\Gamma \in L} \sigma(\Gamma) = \sigma(Cyl(\mathcal{A}/\mathcal{G}))$$

where the projective limit is referred to the family $\{\sigma(\Gamma), \pi_{\Gamma\Gamma'}, L\}$, with $\pi_{\Gamma\Gamma'} := {}^t i_{\Gamma'\Gamma}|_{\sigma(\Gamma')}$.

The last step before the theorem of characterization of $\overline{\mathcal{A}/\mathcal{G}}$ as a projective limit is given by the next theorem.

Theorem 4.11.1 *The following assertions hold.*

1. *Let φ be a continuous linear functional on $Hol(M, G)$, then its restriction $\varphi_\Gamma := \varphi|_{A(\Gamma)}$ is a continuous linear functional on $A(\Gamma)$ and the family $\{\varphi_\Gamma\}(\Gamma \in L)$ satisfies the following conditions:*

   $$i)\ \varphi_{\Gamma'}|_{A(\Gamma)} = \varphi_\Gamma \qquad \forall \Gamma' \geq \Gamma;$$

   ii) the collection $\{\|\varphi_\Gamma\|\}(\Gamma \in L)$ admits a finite maximum.

   The property i) is called **consistency**, *the property ii) instead expresses the* **uniform boundness** *of the family $\{\varphi_\Gamma\}(\Gamma \in L)$;*

2. *If φ is a state of $Hol(M, G)$, then φ_Γ is a state of $A(\Gamma)$, for every $\Gamma \in L$;*

3. *If φ is a character of $Hol(M, G)$, then φ_Γ is a character of $A(\Gamma)$, for every $\Gamma \in L$;*

4. By converse, a family $\{\varphi_\Gamma\}(\Gamma \in L)$ of continuous linear functionals (resp. states, characters) of the C^*-algebras $A(\Gamma)$ satisfying the conditions i) and ii) of 1. defines a continuous linear functional φ (resp. a state, a character) on $Hol(M,G)$ whose restriction to $A(\Gamma)$ is precisely φ_Γ, for every $\Gamma \in L$.

Proof. 1. The consistency is obvious, the uniform boundness is shown simply by the observation that for every $\Gamma \in L$ one has:

$$\|\varphi_\Gamma\| = \sup_{f \in A(\Gamma), \|f\|=1} |\varphi(f)| \leq \|\varphi\| < +\infty.$$

2. The norm of the state φ is $\|\varphi\| = \varphi(\mathbf{1}) = 1$, thus every restriction φ_Γ is a positive linear functional on $A(\Gamma)$ and also $\varphi_\Gamma(\mathbf{1}) = \varphi(\mathbf{1}) = 1$, i.e. φ_Γ is a state of $A(\Gamma)$.

3. Obvious, the characters are the multiplicative states.

4. Consider first the polynomial algebra $\mathcal{W}$ generated by the Wilson functions and define on it the functional:

$$\begin{array}{rccc} \varphi_0 : & \mathcal{W} & \longrightarrow & \mathbb{C} \\ & p & \mapsto & \varphi_0(p) := \varphi_\Gamma(p) \end{array}$$

where Γ is any graph containing all the images of the loops which label the Wilson functions generating the polynomial

$$p = p_0 + \sum_{j_1} p_{j_1} T_{\alpha_{j_1}} + \sum_{j_2} p_{j_1,j_2} T_{\alpha_{j_1}} T_{\alpha_{j_2}} + \dots .$$

The functional φ_0 is *well defined*, in fact, thanks to the consistency of the family $\{\varphi_\Gamma\}(\Gamma \in L)$, if Γ' is another graph containing the images of the loops above, then $\varphi_\Gamma(p) = \varphi_{\Gamma\cap\Gamma'}(p) = \varphi_{\Gamma'}(p)$.

φ_0 is *linear*, in fact if $p, q \in \mathcal{W}$ then it certainly exists a graph Γ such that $p + q \in A(\Gamma)$, hence, thanks to the linearity of φ_Γ on $A(\Gamma)$:

$$\begin{cases} \varphi_0(p+q) = \varphi_\Gamma(p+q) = \varphi_\Gamma(p) + \varphi_\Gamma(q) = \varphi_0(p) + \varphi_0(q); \\ \varphi_0(\lambda p) = \varphi_\Gamma(\lambda p) = \lambda\varphi_\Gamma(p) = \lambda\varphi_0(p), \ \lambda \in \mathbb{C}. \end{cases}$$

φ_0 is *bounded*, this follows from the boundness of φ_Γ:

$$|\varphi_0(p)| = |\varphi_\Gamma(p)| \leq \|\varphi_\Gamma\|\|p\|$$

for every $p \in \mathcal{W}$.

Being $\mathcal{W}$ dense in $Hol(M,G)$, thanks to the theorem of bounded extension of bounded linear functionals, φ_0 can be extended to a unique bounded linear functional φ on $Hol(M,G)$ whose restriction to every $A(\Gamma)$ is φ_Γ.

To prove that φ is a state when the functionals φ_Γ are states it is previously necessary to observe that if the functionals φ_Γ are positive for every Γ then also φ is positive. In fact, since $\mathcal{W} \subset Cyl(\mathcal{A}/\mathcal{G})$, then $\varphi(p) \geq 0$ for every $p \geq 0$, $p \in \mathcal{W}$, thanks to the positivity of the states φ_Γ. Moreover every $f \in Hol(M,G)$, $f \geq 0$, is the uniform limit of a sequence of positive polynomials, this is easy to verify by taking the square root $\sqrt{f}$ of f (it certainly exists because f is positive!) and by writing it as the uniform limit of Wilson polynomials: $\sqrt{f} = \lim_n p_n$. By definition of square root, $f = \lim_n {p_n}^* p_n$ and ${p_n}^* p_n \geq 0$ for every $n \in \mathbb{N}$, thus $\varphi(f) = \lim_n \varphi({p_n}^* p_n) \geq 0$ by virtue of the theorem of persistence of the signum.

Now, if $\{\varphi_\Gamma\}(\Gamma \in L)$ is a family of states of the C^*-algebras $A(\Gamma)$, then φ is a state of $Hol(M,G)$, in fact the states are positive and so φ is positive (for what just shown), hence its norm is the value assumed in the unit element of the algebra: $\|\varphi\| = \varphi(\mathbf{1}) = \varphi_\Gamma(\mathbf{1}) = 1$, for all $\Gamma \in L$.

Finally if the functionals φ_Γ are characters of the algebras $A(\Gamma)$, then φ_0 is multiplicative on $\mathcal{W}$ and so, written the functions $f, g \in Hol(M,G)$ as $f = \lim_n p_n$, $g = \lim_n q_n$, with $\{p_n\}, \{q_n\}(n \in \mathbb{N}) \subset \mathcal{W}$, one has:

$$\varphi(fg) = \lim_n \varphi_0(p_n q_n) = \lim_n \varphi_0(p_n)\varphi_0(q_n) = \lim_n \varphi(p_n)\varphi(q_n)$$

because φ and φ_0 act in the same way on $\mathcal{W}$.

Thanks to the continuity of φ it follows that:

$$\varphi(fg) = \varphi(\lim_n p_n)\,\varphi(\lim_n q_n) = \varphi(f)\varphi(g)$$

showing that φ is a character. □

Theorem 4.11.2 (Projective characterization of $\overline{\mathcal{A}/\mathcal{G}}$) *The spectrum of $Hol(M,G)$ is homeomorphic to the projective limit of the family $\{\sigma(\Gamma), \pi_{\Gamma\Gamma'}, L\}$ w.r.t. the initial topology:*

$$\overline{\mathcal{A}/\mathcal{G}} \simeq \varprojlim_{\Gamma\in L} \sigma(\Gamma)\,.$$

Proof. First of all observe that there is a set-theoretical bijection between $\overline{\mathcal{A}/\mathcal{G}}$ and $\varprojlim_{\Gamma\in L}\sigma(\Gamma)$ because $Cyl(\mathcal{A}/\mathcal{G})$ is dense in $Hol(M,G)$, hence the characters of $Cyl(\mathcal{A}/\mathcal{G})$ can be univocally extended to characters of $Hol(M,G)$ and, by converse, the characters of $Hol(M,G)$ reduces to characters of $Cyl(\mathcal{A}/\mathcal{G})$ simply by restriction, thus these C^*-algebras have the same spectrum.

But then, from 3.and 4.of the previous theorem and from theorem 4.10.2, it follows that $\varprojlim_{\Gamma\in L}\sigma(\Gamma) = \sigma(Cyl(\mathcal{A}/\mathcal{G})) = \sigma(Hol(M,G))$, set-theoretically.

Since the Wilson functions generate $Hol(M,G)$, the topology on its spectrum $\overline{\mathcal{A}/\mathcal{G}}$ is the initial topology defined as the smallest topology in which the (Gelfand transformed of the) Wilson functions are continuous.

For the same reason the spectra $\sigma(\Gamma)$ have the initial topology defined as the smallest topology in which the Wilson functions T_α, with $\alpha^* \subset \Gamma$, are continuous.

By remembering that a projective limit of compact Hausdorff spaces inherits the initial topology the theorem follows. □

Pictorially, the duality between the inductive family of C^*-algebras $A(\Gamma)$ and the projective family of their spectra $\sigma(\Gamma)$ can be represented as follows:

$$\ldots \subseteq A(\Gamma) \subseteq \ldots \subseteq A(\Gamma') \subseteq \ldots \longrightarrow \varinjlim_{\Gamma\in L} A(\Gamma) \equiv Cyl(\mathcal{A}/\mathcal{G})$$

$$\ldots \supseteq \sigma(\Gamma) \supseteq \ldots \supseteq \sigma(\Gamma') \supseteq \ldots \longleftarrow \varprojlim_{\Gamma\in L} \sigma(\Gamma) \equiv \overline{\mathcal{A}/\mathcal{G}}\,.$$

4.11.1 The characterization of the spectra $\sigma(\Gamma)$

The spectra $\sigma(\Gamma)$ can be explicitly characterized in a useful way by using a few topological results.

Let Γ be a *connected graph* and $\star$ a fixed point of Γ, then its **fundamental group**, or **first homotopy group**, $\pi_1(\Gamma,\star)$, is the group of the homotopy classes of loops based on $\star$ with image contained in Γ.

Since Γ is assumed to be connected, the choice of the (fixed) base-point $\star$ is irrelevant in the definition of the fundamental group, thus I shell denote it simply by $\pi_1(\Gamma)$.

It is well known (see for instance ch.14 of [41]) that $\pi_1(\Gamma)$ *is a free group with* n_Γ *generators*, where n_Γ is the **connectivity** of Γ, i.e. the integer:

$$n_\Gamma = E_\Gamma - V_\Gamma + 1$$

being E_Γ the number of edges of Γ and V_Γ the number of its vertexes; n_Γ *is a topological invariant* of the graph Γ which represents the highest number of edges that can be deleted from the graph without it fails to be connected.

Denote with $L_\star(\Gamma)$ the subgroup of $L_\star(M)$ given by the loops containing at least a representative α with $\alpha^* \subset \Gamma$.

Theorem 4.11.3 *For every graph* Γ *the following assertions hold.*

1. *The group* $L_\star(\Gamma)$ *is isomorphic to* $\pi_1(\Gamma)$*;*
2. *The generators* $\beta_1,\ldots,\beta_{n_\Gamma}$ *of* $\pi_1(\Gamma)$ *form an independent family of loops in* $L_\star(M)$*;*

3. For every fixed graph Γ *the following spaces are homeomorphic:*

$$\sigma(\Gamma) \simeq Hom(L_\star(\Gamma), G)/Ad_G \simeq G^{n_\Gamma}/Ad_G\,.$$

The proof of this theorem can be found in [1], here the only important thing is that the explicit form of the isomorphism in 3. is given by the following map:

$$\begin{array}{rccc} \phi_{\vec{\beta}}: & G^{n_\Gamma}/Ad_G & \longrightarrow & Hom(L_\star(\Gamma), G)/Ad_G \\ & [g_1, \ldots, g_{n_\Gamma}] & \mapsto & \phi_{\vec{\beta}}([g_1, \ldots, g_{n_\Gamma}]) := [H_{\vec{g}}] \end{array}$$

where $\vec{\beta} \equiv \{\beta_1, \ldots, \beta_{n_\Gamma}\}$ is a fixed family of generators of $\pi_1(\Gamma)$ and $H_{\vec{g}}(\beta_i) := g_i$, $i = 1, \ldots, n_\Gamma$, $\vec{g} = (g_1, \ldots, g_{n_\Gamma})$. It is easy to see that $\phi_{\vec{\beta}}$ is invertible and that its inverse is the evaluation map in the generators of $\pi_1(\Gamma)$, i.e. $\phi_{\vec{\beta}}^{-1} = ev(\beta_1, \ldots, \beta_{n_\Gamma})$.

Since the evaluation map is certainly continuous, the only thing that remains to do is to prove that $\phi_{\vec{\beta}}$ is continuous.

Remember that, if $[\tilde{H}]$ is a fixed element of $Hom(L_\star(\Gamma), G)/Ad_G$, then a base of open neighborhoods of $[\tilde{H}]$ is given by:

$$U := \{[H] \ : \ \frac{1}{N}|Tr(H(\alpha_i)) - Tr(\tilde{H}(\alpha_i))| < \varepsilon,\ i = 1, \ldots, k\}$$

for a given finite set of loops α_i, $i = 1, \ldots, k$.

The proof of the continuity of $\phi_{\vec{\beta}}$ is equivalent to the proof that $\phi_{\vec{\beta}}^{-1}(U)$ is open.

This fact certainly holds when $\alpha_i = \beta_i$, $i = 1, \ldots, k$, in fact in this situation $Tr(H_{\vec{g}}(\alpha_i)) = Tr(H_{\vec{g}}(\beta_i)) = Tr(g_i)$, and, being Tr a continuous function, the set of the $(g_1, \ldots, g_n)$ such that $|Tr(g_i) - \lambda_i| < \varepsilon$ is an open set in G^{n_Γ} for every $\lambda_i \in \mathbb{C}$.

Now decompose every α_i as

$$\alpha_i = {\beta_1}^{m^i_{1,1}} \cdots {\beta_{n_\Gamma}}^{m^i_{n_\Gamma,1}} {\beta_1}^{m^i_{1,2}} \cdots {\beta_{n_\Gamma}}^{n^i_{n_\Gamma,2}} \cdots$$

then

$$Tr(H_{\vec{g}}(\alpha_i)) = Tr({g_1}^{m^i_{1,1}} \cdots {g_{n_\Gamma}}^{m^i_{n_\Gamma,1}} {g_1}^{m^i_{1,2}} \cdots {g_{n_\Gamma}}^{n^i_{n_\Gamma,2}} \cdots)$$

since $H_{\vec{g}}$ is a homomorphism and so factorizes as the loops.

The continuity of the trace implies again that the set of the $(g_1, \ldots, g_{n_\Gamma})$ such that $|Tr({g_1}^{m^i_{1,1}} \cdots {g_{n_\Gamma}}^{m^i_{n_\Gamma,1}} {g_1}^{m^i_{1,2}} \cdots {g_{n_\Gamma}}^{n^i_{n(\Gamma,2)}} \cdots) - \lambda_i| < \varepsilon$ is open in G^{n_Γ} for every $\lambda_i \in \mathbb{C}$.

Hence, taking in particular $\lambda_i = Tr(\tilde{H}(\alpha_i))$, for every $i = 1, \ldots, k$, one has the thesis.

Finally, the homeomorphism between $Hom(L_\star(\Gamma), G)/Ad_G$ and $\sigma(\Gamma)$ is just a consequence of the results discussed in 4.4 applied to the C^*-algebras $A(\Gamma)$ instead of the holonomy C^*-algebra $Hol(M, G)$. $\square$

As an immediate corollary of the previous theorem one obtains this explicit and useful projective characterization of $\overline{\mathcal{A}/\mathcal{G}}$:

$$\overline{\mathcal{A}/\mathcal{G}} \simeq \varprojlim_{\Gamma \in L} G^{n_\Gamma}/Ad_G \, .$$

4.12 Cylindrical measures on $\overline{\mathcal{A}/\mathcal{G}}$

The references for this sections are [63] and [97].

First of all remember that a probability space is a triple (Ω, Σ, p), where Ω is a non-empty measurable space, Σ is a σ-algebra on Ω and p is a probability measure on Ω: $p(\Omega) = 1$.

If T is any index set, one can consider the family $\{(\Omega_t, \Sigma_t, p_t\}(t \in T)$ of measurable spaces and define the *product σ-algebra* $\prod_{t\in T} \Sigma_t$ on the cartesian product $\prod_{t\in T} \Omega_t$ to be the smallest σ-algebra containing the cylindrical subsets $\prod_{t\in T} M_t \subset \prod_{t\in T} \Omega_t$, where $\prod_{t\in T} M_t$ is said to be cylindrical if

1. $M_t \in \Sigma_t$, $\forall t \in T$;
2. $M_t = \Omega_t$ but for a finite subset of indexes in T.

This σ-algebra is the smallest which makes the projections $p_r : \prod_{t\in T} \Omega_t \to \Omega_r$ measurable maps, for every fixed $r \in T$; one says shortly that this is the σ-algebra generated by the projections.

The probability measure p defined on the cylindrical subsets by:

$$p(\prod_{t\in T} M_t) := \prod_{t\in T} p_t(M_t)$$

is called, for obvious reasons, **cylindrical measure** and it always extends to a measure defined on the entire product σ-algebra.

Now take, in particular, a directed set J and suppose that the family of probability spaces $\{\Omega_j\}(j \in J)$ has measurable projections $\pi_{jj'}$, defined for every $j \leq j'$ and satisfying the axioms of a projective family, then the triple $\{\Omega_j, \pi_{jj'}, J\}$ is said to be a **projective family of probability spaces**.

Suppose now to have a measure μ on the projective limit Ω of this family, then the push-forward of μ via the canonical projection $\pi_j : \Omega \to \Omega_j$, i.e. $\mu_j := {\pi_j}_* \mu \equiv \mu \circ \pi_j$, is a measure on Ω_j, for every $j \in J$.

Furthermore the family of measures $\{\mu_j\}(j \in J)$ satisfies the **consistency condition**

$$\mu_j = (\pi_{jj'})_* \mu_{j'} = \mu_{j'}|_{\Omega_j} \circ \pi_{jj'}$$

which guaranties that there is no ambiguity when a portion of Ω_j is measured directly by μ_j or by the restriction of $\mu_{j'}$ to Ω_j.

A family of measures $\{\mu_j\}(j \in J)$ satisfying the consistency condition is said to be a **promeasure**.

A classical problem of measure theory is to study when it is possible to construct a measure μ on Ω starting by a promeasure, i.e. when it is possible to obtain a representation theorem for measures on projective limits, since the inverse process is always possible, as just discussed.

When the index set J is numerable this representation theorem is available every time the spaces Ω_j are σ-compact metric spaces and the promeasure is Borel-like.

However the request of numerability of the index set J is quite restrictive, luckily when the probability spaces of the projective family are compact the extension of a promeasure to a measure on the projective limit is always possible.

Before formalizing this assertion in a theorem it is worth remembering once again that, when the spaces Ω_j of a projective family are compact Hausdorff spaces, the projective limit Ω is a non-empty compact Hausdorff space itself; furthermore, in this situation, the algebra of the cylindrical functions on Ω satisfies the hypothesis of the Stone-Weierstrass theorem and so it is dense in the algebra of the continuous complex-valued functions on Ω:

$$\overline{Cyl(\Omega)} = \mathcal{C}(\Omega) \qquad \text{(if } \Omega \text{ is compact).}$$

To simplify the notation in the sequel a regular Borel probability measure will be simply called "probability measure".

Theorem 4.12.1 *Let $\{\Omega_j, \pi_{jj'}, J\}$ be a projective family of compact Hausdorff spaces with projective limit Ω.*

Then there is a bijective correspondence between probability measures on Ω and probability promeasures $\{\mu_j\}(j \in J)$.

All such measures are cylindrical.

Proof. It has to be proved that a probability promeasure $\{\mu_j\}(j \in J)$ univocally defines a probability measures on Ω.

Define the linear functional

$$\begin{array}{rccl} F: & \coprod_{j\in J} \mathcal{C}(\Omega_j) & \longrightarrow & \mathbb{C} \\ & f_j & \mapsto & F(f_j) := \int_{\Omega_j} f_j \, d\mu_j \end{array}$$

$\forall f_j \in \mathcal{C}(\Omega_j)$.

Thanks to the consistency condition of the measures appearing in the promeasure, F factorizes to $Cyl(\Omega)$ in a natural fashion.

Being bounded, F admits a unique extension to a bounded linear functional $\bar{F}$ on the closure of $Cyl(\Omega)$, i.e. on $\mathcal{C}(\Omega)$.

Thanks to the Riesz-Markov theorem it exists a unique probability measure μ on Ω which represents the functional $\bar{F}$ in the usual way, i.e. $\bar{F}(f) := \int_\Omega f\, d\mu$, $\forall f \in \mathcal{C}(\Omega)$.

The measure μ is obviously cylindrical. □

This result can be specialized to the projective family of the compact Hausdorff spaces $\{\sigma(\Gamma)\}(\Gamma \in L)$, which gives rise to the the compact Hausdorff space $\overline{\mathcal{A}/\mathcal{G}}$, to obtain the following important result.

Corollary 4.12.1 *There is a bijection between the probability measures on* $\overline{\mathcal{A}/\mathcal{G}}$ *and the probability promeasures* $\{\mu_\Gamma\}(\Gamma \in L)$ *on the spectra* $\sigma(\Gamma)$.

Thanks to the characterization $\sigma(\Gamma) \simeq G^{n_\Gamma}/Ad_G$ an explicit (and natural) promeasure which gives rise to a probability measure on $\overline{\mathcal{A}/\mathcal{G}}$ is given by the family of the normalized Haar measures dg^{n_Γ} on the groups G^{n_Γ}, which are Ad_G-invariant (thanks to the assumption of compactness for G) and thus projects unaffected to the quotient G^{n_Γ}/Ad_G.

The probability measure obtained from the promeasure $\{dg^{n_\Gamma}\}(\Gamma \in L)$ is called the **uniform measure** on $\overline{\mathcal{A}/\mathcal{G}}$ and denoted by μ_0.

If a function $f \in Hol(M, G) \simeq \mathcal{C}(\overline{\mathcal{A}/\mathcal{G}})$ is cylindrical w.r.t. the index-graph Γ, i.e. it exists $f_\Gamma \in \mathcal{C}(G^{n_\Gamma}/Ad_G)$ such that $f = f_\Gamma \circ \pi_\Gamma$, then its explicit integral w.r.t. the uniform measure is given by:

$$\int_{\overline{\mathcal{A}/\mathcal{G}}} f(\bar{A})\, d\mu_0(\bar{A}) = \int_{G^{n_\Gamma}} f_\Gamma(g_1, \ldots, g_{n_\Gamma})\, dg^{n_\Gamma}(g_1, \ldots, g_{n_\Gamma})\,.$$

Thanks to the density of $Cyl(\overline{\mathcal{A}/\mathcal{G}})$ in $\mathcal{C}(\overline{\mathcal{A}/\mathcal{G}})$, the formula above extends (by uniform limit) to all the functions of $\mathcal{C}(\overline{\mathcal{A}/\mathcal{G}})$.

J.Baez has found in [15] a way to construct (in general non-faithful) measures on $\overline{\mathcal{A}/\mathcal{G}}$ starting from random variables, his work contemplates the uniform measure as a particular case.

4.12.1 Diffeomorphism invariant measures on $\overline{\mathcal{A}/\mathcal{G}}$

Now I put the attention on the invariance of the measures on $\overline{\mathcal{A}/\mathcal{G}}$ under diffeomorphisms.

Let $\text{Diff}_0(M)$ be the group of the *analytical diffeomorphisms of M homotopic to the identity map*.

The action of $\text{Diff}_0(M)$ on $Hol(M,G)$ is defined in the following way:

$$\begin{array}{ccc} Hol(M,G) \times \text{Diff}_0(M) & \longrightarrow & Hol(M,G) \\ (T_\alpha, \Psi) & \mapsto & \Psi_* T_\alpha, \end{array}$$

$(\Psi_* T_\alpha)([A]) := T_{\Psi \circ \alpha}([A])$, for every $[A] \in \mathcal{A}/\mathcal{G}$.

This action is well defined because the Wilson functions generate $Hol(M,G)$.

Observe that $\Psi \circ \alpha$ is precisely the loop α deformed by the diffeomorphism Ψ.

It is useful to reformulate this action using the structure of principal fiber bundle.

In the chapter 1. we have seen that there is an injective homomorphism

$$\begin{array}{cccc} \flat : & \text{Aut}(P) & \longrightarrow & \text{Diff}(M) \\ & \Phi & \mapsto & \flat(\Psi) = \Psi \end{array}$$

whose range contains $\text{Diff}_0(M)$, thus the action of $\text{Diff}_0(M)$ on the Wilson functions can be reformulated as follows

$$\Psi_* T_\alpha([A]) = T_{\Psi \circ \alpha}([A]) = T_\alpha(\Phi^*([A]))$$

where Φ^* is the pull-back of the automorphism $\Phi \in \text{Aut}(P)$ such that $\flat(\Phi) = \Psi$.

This action naturally extend to the Wilson polynomials p by the position $\Psi_* p([A]) := p(\Psi^*([A]))$.

Observe that the deformation of the loop α doesn't affect the norm of the Wilson functions, since the holonomy of the loop α is invariant under the diffeomorphisms of $\text{Diff}_0(M)$. This property also extends to the Wilson polynomials: $\|\Psi_* p\| = \|p\|$.

Since the Wilson polynomials are dense in $Hol(M,G)$, the action of $\text{Diff}_0(M)$ can be extended to an isometric action on the entire $Hol(M,G)$, i.e., using the same symbol

$$\begin{array}{cccc} \Psi_* : & Hol(M,G) & \longrightarrow & Hol(M,G) \\ & f & \mapsto & \Psi_* f := \lim_n \Psi_* p_n \end{array}$$

where $\{p_n\}(n \in \mathbb{N}) \subset \mathcal{W}$, $f = \lim_n p_n$.

This fact gives the possibility to define a representation of $\text{Diff}_0(M)$ in isometries of $Hol(M,G)$:

$$\begin{array}{cccc} \rho : & \text{Diff}_0(M) & \longrightarrow & \text{Aut}(Hol(M,G)) \\ & \Psi & \mapsto & \rho(\Psi) := \Psi_*. \end{array}$$

Def. 4.12.1 *A positive linear functional* $\varphi \in Hol(M,G)^*$ *is said to be* **invariant under diffeomorphisms** *if*

$$\varphi(\Psi_* f) = \varphi(f) \qquad \forall f \in Hol(M,G),\ \forall \Psi \in \mathit{Diff}_0(M).$$

A probability measure on $\overline{\mathcal{A}/\mathcal{G}}$ *is called invariant under diffeomorphisms if its corresponding state* φ_μ *possesses this invariance.*

If $\Psi_*\mu$ denotes the measure corresponding to the functional $\varphi_\mu \circ \Psi_*$ then the invariance of the measure μ is written symbolically as: $\Psi_*\mu = \mu$.

Since $\overline{\mathcal{A}/\mathcal{G}}$ is compact, fixed a positive regular Borel measure μ on it, one has that $L^2(\overline{\mathcal{A}/\mathcal{G}},\mu) \subset L^1(\overline{\mathcal{A}/\mathcal{G}},\mu)$, hence the Radon-Nykodim theorem implies that, for every $f \in L^2(\overline{\mathcal{A}/\mathcal{G}},\mu)$, there exists one and only one measure μ_f which is absolutely continuous w.r.t. μ and such that f can be written as the Radon-Nykodim derivative: $f = \frac{d\mu_f}{d\mu}$.

This enables to construct a unitary representation of $\mathrm{Diff}_0(M)$ supported by $L^2(\overline{\mathcal{A}/\mathcal{G}},\mu)$ and labelled by the positive regular Borel measure μ, this representation is defined by:

$$\begin{array}{rccl} U: & \mathrm{Diff}_0(M) & \longrightarrow & \mathcal{U}(L^2(\overline{\mathcal{A}/\mathcal{G}},\mu)) \\ & \Psi & \mapsto & U_\Psi \end{array}$$

where

$$\begin{array}{rccl} U_\Psi: & L^2(\overline{\mathcal{A}/\mathcal{G}},\mu) & \longrightarrow & L^2(\overline{\mathcal{A}/\mathcal{G}},\mu) \\ & f & \mapsto & U_\Psi(f) := \Psi_*\mu_f. \end{array}$$

An element $f \in L^2(\overline{\mathcal{A}/\mathcal{G}},\mu)$ will be called invariant under diffeomorphisms if the corresponding measure μ_f has this property, i.e. if $\Psi_*\mu_f = \mu_f$, for every $\Psi \in \mathrm{Diff}_0(M)$.

Finally consider again the family of functionals $\{\varphi_\Gamma\}(\Gamma \in L)$ of the previous section and define the so-called **covariance condition** as:

$$\varphi_{\Gamma'} \circ \Psi_* = \varphi_\Gamma \qquad \text{whenever } \varphi(\Gamma) = \Gamma'.$$

As the same, one says that the family of probability measures $\{\mu_\Gamma\}(\Gamma \in L)$ satisfies the covariance condition if the corresponding family of functionals does.

It is straightforward to see that if the condition of covariance is satisfied by a promeasure, then the cylindrical measure induced on $\overline{\mathcal{A}/\mathcal{G}}$ is invariant under diffeomorphisms. This assertion is formalized in the next theorem.

Theorem 4.12.2 *There is a bijection between the diffeomorphism invariant probability measures on* $\overline{\mathcal{A}/\mathcal{G}}$ *and the probability promeasures* $\{\mu_\Gamma\}(\Gamma \in L)$ *satisfying the covariance condition.*

I stress that

the uniform measure μ_0 is invariant under diffeomorphisms.

In fact the only possible dependence of μ_0 on the diffeomorphisms of M is contained in the connectivity n_Γ, but this is a topological invariant and so it is unaffected by them, hence the covariance condition is automatically satisfied.

Furthermore note that μ_0 **is also gauge-invariant** since the Haar measures on the compact gauge groups are bi-invariant.

The properties of μ_0 has been widely studied by D.Marolf and J.Mourão in [66]; the most remarkable results obtained in that work are the following:

- μ_0 **is faithful**, i.e. $f \in \mathcal{C}(\overline{\mathcal{A}/\mathcal{G}})$, $f \geq 0$ and $\int_{\overline{\mathcal{A}/\mathcal{G}}} f \, d\mu_0 = 0$ implies $f \equiv \mathbf{0}$;

- μ_0 **is concentrated on the generalized connections**, i.e.

$$\begin{cases} \mu_0(\mathcal{A}/\mathcal{G}) = 0; \\ \mu_0(\overline{\mathcal{A}/\mathcal{G}}) = 1. \end{cases}$$

4.13 Alternative construction of $\overline{\mathcal{A}/\mathcal{G}}$ and $L^2(\overline{\mathcal{A}/\mathcal{G}})$

We want to present another, equivalent, construction of the quantum configuration space $\overline{\mathcal{A}/\mathcal{G}}$. Two very clear references are [37] and [90].

The algebraic characterization of $\overline{\mathcal{A}/\mathcal{G}}$ shows that one of its fundamental characteristic is that the connections and the gauge transformations that gives rise to this space are 'generalized' in the sense that they are objects satisfying only the algebraic rules of connection and gauge transformations and dropping out all the topological and differential ones.

Remembering that a (smooth) connection A is uniquely determined by its parallel transport (or holonomy map), i.e. by a (smooth) homomorphism H_A from the groupoid *Path* of piecewise analytic paths in M into the gauge group G, it is quite natural define a *generalized connection* simply as such a homomorphism, but without the smoothness condition!

Thus the space of generalized connections becomes

$$\overline{\mathcal{A}} := Hom(Path, G).$$

The very important fact is that, whenever G is a semisimple compact connected Lie group, $\mathcal{A}$ is densely embedded in $\overline{\mathcal{A}}$ and the latter space can be endowed with a natural measure.

In fact also $\overline{\mathcal{A}}$ is homeomorphic to a projective limit:

$$\overline{\mathcal{A}} \simeq \varprojlim_{\Gamma} \overline{\mathcal{A}}_\Gamma, \quad \overline{\mathcal{A}}_\Gamma := Hom(Path_\Gamma, G)$$

where Γ is a graph in M and $Path_\Gamma$ is the subgroupoid of $Path$ freely generated by the edges of Γ.

It turns out that, for every graph Γ with E_Γ edges and V_Γ vertexes, $\overline{\mathcal{A}}_\Gamma$ is homeomorphic with G^{E_Γ} via the evaluation map in the edges of the graph:

$$\begin{array}{rccl} ev_{e_1,\dots,e_{E_\Gamma}}: & Hom(Path_\Gamma, G) & \longrightarrow & G^{E_\Gamma} \\ & H_\Gamma & \mapsto & ev_{e_1,\dots,e_{E_\Gamma}}(H_\Gamma) = (H_\Gamma(e_1), \dots, H_\Gamma(e_{E_\Gamma})). \end{array}$$

A generic element of $\overline{\mathcal{A}}$ will be indicated with $\bar{A}$ and one of $\overline{\mathcal{A}}_\Gamma$ with $\bar{A}_\Gamma$.

By means of the projective characterization of $\overline{\mathcal{A}}$ and of the identification $\overline{\mathcal{A}}_\Gamma \simeq G^{E_\Gamma}$, a natural probability measure on $\overline{\mathcal{A}}$ is constructed by taking the projective limit of the normalized Haar measures appearing at every step of the limit. The resulting measure is called again *uniform measure* and indicated with μ_0.

Now, remembering that the set of gauge transformations of a trivial principal bundle is isomorphic to $\mathcal{C}^\infty(M, G)$ and remembering also that canonical loop quantum gravity deals with $SU(2)$-principal bundles over a 3-dimensional base space, which are all trivial as stated in chapter 1, one defines the set of the *generalized gauge transformations* of the principal fiber bundle $P(M, G)$ as the set of *all* maps from M to G, i.e.

$$\overline{\mathcal{G}} := Maps(M, G) \equiv G^M.$$

It can be shown that $\overline{\mathcal{G}}$ is homeomorphic to the following projective limit:

$$\overline{\mathcal{G}} \simeq \varprojlim_{\Gamma} \overline{\mathcal{G}}_\Gamma, \quad \overline{\mathcal{G}}_\Gamma := Maps(V_\Gamma, G) \equiv G^{V_\Gamma}.$$

A generic element of $\overline{\mathcal{G}}$ will be denoted with $\bar{g}$, and one of $\overline{\mathcal{G}}_\Gamma$ with $\bar{g}_\Gamma$.

If $g(\gamma_t)$ denotes the element of G individuated by the parallel transport $H_A : Path \to G$ relative to the connection $A \in \mathcal{A}$ and calculated in the point γ_t, then the parallel transport $H_{A'}$ relative to a connection $A' \in \mathcal{A}$, obtained by A after a gauge transformation, is related to H_A by the following expression:

$$H_{A'}(\gamma) = g(\gamma_0)^{-1} H_A(\gamma) g(\gamma_1) \qquad \forall \gamma \in Path.$$

Then, fixed a graph Γ, it is natural to define the (right) action of $\overline{\mathcal{G}}_\Gamma$ on $\overline{\mathcal{A}}_\Gamma$, as:

$$\begin{array}{ccc} \overline{\mathcal{A}}_\Gamma \times \overline{\mathcal{G}}_\Gamma & \longrightarrow & \overline{\mathcal{A}}_\Gamma \\ (\bar{A}_\Gamma, \bar{g}_\Gamma) & \mapsto & \bar{A}_\Gamma \circ \bar{g}_\Gamma, \end{array}$$

$(\bar{A}_\Gamma \circ \bar{g}_\Gamma)(\gamma) := \bar{g}_\Gamma(\gamma_0)^{-1}\bar{A}_\Gamma(\gamma)\bar{g}_\Gamma(\gamma_1)$, for every $\gamma \in Path_\Gamma$, and extend it to a right action of $\overline{\mathcal{G}}$ on $\overline{\mathcal{A}}$ by using their projective characterizations, i.e.

$$\begin{array}{rcl} \overline{\mathcal{A}} \times \overline{\mathcal{G}} & \longrightarrow & \overline{\mathcal{A}} \\ (\bar{A}, \bar{g}) & \mapsto & \bar{A} \circ \bar{g}, \end{array}$$

$(\bar{A} \circ \bar{g})(\gamma) := \{(\bar{A}_\Gamma \circ \bar{g}_\Gamma)(\gamma)\}_\Gamma$.

By defining $\overline{\mathcal{A}/\mathcal{G}}_\Gamma := \overline{\mathcal{A}}_\Gamma/\overline{\mathcal{G}}_\Gamma$, it can be shown that the space of the generalized connections modulo generalized gauge transformations, i.e. $\overline{\mathcal{A}}/\overline{\mathcal{G}}$, can be homeomorphically characterized as follows:

$$\overline{\mathcal{A}}/\overline{\mathcal{G}} \simeq \varprojlim_\Gamma \overline{\mathcal{A}/\mathcal{G}}_\Gamma \equiv \overline{\mathcal{A}/\mathcal{G}}.$$

The result of dense injection of $\mathcal{A}$ in $\overline{\mathcal{A}}$ for compact connected semisimple G can be easily extended to the space of the gauge orbits, i.e. $\mathcal{A}/\mathcal{G}$ is densely embedded in $\overline{\mathcal{A}/\mathcal{G}}$.

Moreover, since the generalized gauge transformations act as translations on the generalized connections and since the normalized Haar measure of a compact group is invariant under translations, the uniform measure μ_0 on $\overline{\mathcal{A}}$ projects naturally on $\overline{\mathcal{A}/\mathcal{G}}$.

The Hilbert space $L^2(\overline{\mathcal{A}/\mathcal{G}})$ with respect to the uniform measure can be also viewed as the subspace of the elements of $L^2(\overline{\mathcal{A}})$ that are invariant under the induced action of $\overline{\mathcal{G}}$ on it.

This action is defined, for every Γ, as:

$$\begin{array}{rcl} \overline{\mathcal{G}}_\Gamma \times L^2(\overline{\mathcal{A}}_\Gamma) & \longrightarrow & L^2(\overline{\mathcal{A}}_\Gamma) \\ (\bar{g}_\Gamma, \psi_\Gamma) & \mapsto & \bar{g}_\Gamma \circ \psi_\Gamma, \end{array}$$

$(\bar{g}_\Gamma \circ \psi_\Gamma)(\bar{A}_\Gamma) := \psi_\Gamma(\bar{g}_\Gamma^{-1} \circ \bar{A}_\Gamma)$ (the inversion of $\bar{g}_\Gamma$ is necessary to obtain a left action from the right action of $\overline{\mathcal{G}}_\Gamma$ on $\overline{\mathcal{A}}_\Gamma$).

The action on the projective limits is obviously defined as follows:

$$\begin{array}{rcl} \overline{\mathcal{G}} \times L^2(\overline{\mathcal{A}}) & \longrightarrow & L^2(\overline{\mathcal{A}}) \\ (\{\bar{g}_\Gamma\}_\Gamma, \{\psi_\Gamma\}_\Gamma) & \mapsto & \{\bar{g}_\Gamma \circ \psi_\Gamma\}_\Gamma. \end{array}$$

To this action there corresponds a unitary representation of $\overline{\mathcal{G}}$ supported on $L^2(\overline{\mathcal{A}})$ in the usual way and it can be easily seen that the subspace of $L^2(\overline{\mathcal{A}})$ invariant under this action can be identified with $L^2(\overline{\mathcal{A}/\mathcal{G}})$.

4.14 The algorithm of the loop quantization

One of the greatest obstacle in the canonical quantization of a gauge theory on a general spacetime is the absence of a measure on the classical configuration space of the theory itself. The reason is that this space, typically denoted with $\mathcal{A}/\mathcal{G}$, is given by (smooth) connections modulo (smooth) gauge transformations on the principal fiber bundle underlying the theory, this one is an infinite-dimensional non-linear space with a highly non-trivial topological structure, so that one cannot apply the known procedures to endow it with a genuine measure.

The loop quantization overcomes this problem by densely embedding $\mathcal{A}/\mathcal{G}$ in a compact space, denoted with $\overline{\mathcal{A}/\mathcal{G}}$, which is composed by generalized connections modulo generalized gauge transformations. These generalized objects are obtained by dropping out all the topological and differential regularity required on the connections and the gauge transformations and by maintaining only the algebraic ones.

The convenience of this embedding is that the space $\overline{\mathcal{A}/\mathcal{G}}$, unlike $\mathcal{A}/\mathcal{G}$, can be endowed with a natural probability measure, the so-called uniform measure μ_0, which turns out to be invariant both under gauge transformations and under diffeomorphisms of the base space M.

For this and other reasons, the space $\overline{\mathcal{A}/\mathcal{G}}$ is taken to be the quantum configuration space of a gauge theory. Consequently, the Hilbert space $L^2(\overline{\mathcal{A}/\mathcal{G}}, \mu_0)$ is a natural candidate to play the role of the quantum state space.

The loop canonical quantization is a program of **non-perturbative quantization** of gauge theories. This program is quite recent and it is still far from having a complete formulation, in particular at a dynamical level.

The first important assumption in the program relies in the choice of a manifestly gauge-invariant description of gauge theories: the configuration space of these theories is taken to be $\mathcal{A}/\mathcal{G}$ and not $\mathcal{A}$.

With this choice the invariance under gauge transformations is solved at a classical level, being encoded in the configuration space itself, and it doesn't need to be implemented as a constraint in the quantum theory.

However this advantage is paid by a remarkable complication in the mathematical structure: $\mathcal{A}/\mathcal{G}$ is a non-flat space with a highly non-trivial topology, for many interesting theories it *isn't* a manifold and, most important, there are several obstructions to construct measures on it.

These problems explain why, in the usual Hamiltonian quantization of gauge theories, the configuration space is always taken to be $\mathcal{A}$. The invariance under gauge transformations is then introduced as a constraint (**the Gauss constraint**) by means of several procedures as ghosts, gauge fixing, projections and so on, see [48] for a more complete discussion on these topics.

Nevertheless the lattice gauge theory suggested a way to avoid these problems by using the techniques related to the Wilson functions, this suggestion is sensed because, as shown in chapter 4, the gauge invariant information of the connections is fully encoded in the Wilson functions, hence one is naturally leaded to assume **the holonomy algebra as the algebra of the classical observables of gauge theories**.

The quantization is then performed by means of the C^*-algebraic formalism: the self-adjoint elements of the holonomy algebra are promoted to self-adjoint linear operators on a certain Hilbert space containing the **kinematical states** of the quantum theory.

About this Hilbert space it is useful to remember that what usually happens in the quantization of gauge theories (see [45] or [66]) is that on the classical configuration space, denoted generically with X, there is a cylindrical but not σ-additive measure μ which enables to construct the pre-Hilbert space $L^2_{cyl}(X,\mu)$ of the square-integrable cylindrical functions on X; if μ admits an extension to a Borel measure $\bar{\mu}$ on X then the completion of $L^2_{cyl}(X,\mu)$ leads to the Hilbert space $L^2(X,\bar{\mu})$.

However, if this extension is not available, the quantum theory is implemented by extending (on the base of physical and/or mathematical considerations) the classical configuration space X to a wider space $\bar{X}$ on which a genuine measure ν is available, in order to construct the Hilbert space $L^2(\bar{X},\nu)$.

The space $\bar{X}$ is called **the quantum configuration space** and the Hilbert space $L^2(\bar{X},\nu)$ is taken to be **the space of the quantum kinematical states** of the theory.

This is precisely what happens in the loop quantization of gauge theories: the lack of a measure on $\mathcal{A}/\mathcal{G}$ leads to search an extension of this space, the major candidate to the role of quantum configuration space is $\overline{\mathcal{A}/\mathcal{G}}$ for the following reasons:

- first of all $\mathcal{A}/\mathcal{G}$ is injectively and densely embedded in $\overline{\mathcal{A}/\mathcal{G}}$, hence the classical theory is contained in the quantum theory without anomalies;

- $\overline{\mathcal{A}/\mathcal{G}}$ is an infinite-dimensional compact Hausdorff space endowed with a *natural* probability measure, the uniform measure μ_0. Associated to this (faithful) measure there is one and only one faithful representation of the holonomy C^*-algebra $Hol(M,G)$ supported by the Hilbert space $L^2(\overline{\mathcal{A}/\mathcal{G}},\mu_0)$, i.e. the GNS representation:

$$\begin{array}{ccc} Hol(M,G) & \longrightarrow & \mathcal{B}(L^2(\overline{\mathcal{A}/\mathcal{G}},\mu_0)) \\ f & \mapsto & M_{\hat{f}} \end{array}$$

  $M_{\hat{f}}(\psi) := \hat{f}(\bar{A})\psi(\bar{A})$, $\forall \psi \in L^2(\overline{\mathcal{A}/\mathcal{G}},\mu_0)$, $\hat{f} \in \mathcal{C}(\overline{\mathcal{A}/\mathcal{G}}) \subset L^2(\overline{\mathcal{A}/\mathcal{G}},\mu_0)$ is the Gelfand transformed of f. Hence the elements of the holonomy C^*-algebra are promoted to bounded multiplication operators on the Hilbert space $L^2(\overline{\mathcal{A}/\mathcal{G}},\mu_0)$, they are bounded because the Wilson functions (which generate $Hol(M, G)$) are bounded and the Gelfand isomorphism is isometric. The real part of the Wilson functions are thus promoted to bounded self-adjoint operators on $L^2(\overline{\mathcal{A}/\mathcal{G}}, \mu_0)$, i.e. observables in the quantum theory. It is worth noting that the real parts of the Wilson functions generate the same C^*-algebra thanks to the identity ${T_\alpha}^* = T_{\alpha^{-1}}$;

- while the previous are mathematically rigorous motivations for the choice of $\overline{\mathcal{A}/\mathcal{G}}$ as the quantum configuration space, there is a further motivation based on a physical intuition. Since, as proved in chapter 3, gauge-equivalence connections are in one-to-one correspondence to conjugation classes of holonomies, in a lattice gauge theory based on a graph Γ, the configuration space is G^{n_Γ}/Ad_G, hence, being $\overline{\mathcal{A}/\mathcal{G}}$ the projective limit of the family $\{G^{n_\Gamma}/Ad_G\}_\Gamma$, a gauge field theory which has $\overline{\mathcal{A}/\mathcal{G}}$ as quantum configuration space is suitable to be interpreted as the continuous limit of the lattice gauge theories corresponding to every fixed graph, which are approximated (or regularized) theories. The fact that the set of all graphs is closed under diffeomorphisms is of essential importance when the diffeomorphism invariance is taken into account. For the reasons discussed above, a graph Γ is interpreted in the formalism of the loop quantization as a *floating lattice in M*.

The compactification of the configuration space is not a characteristic of this procedure, but it often appears in the quantization of the systems with an infinite number of degrees of freedom, such as field theories. For example in the quantization of the scalar field in d-dimensions the classical configuration space, i.e. the Schwartz space $S(\mathbb{R}^d)$, is substituted by $S'(\mathbb{R}^d)$, the space of the tempered distributions on $\mathbb{R}^d$, in which it is densely embedded.

I stress that the compactification $\mathcal{A}/\mathcal{G} \hookrightarrow \overline{\mathcal{A}/\mathcal{G}}$ is highly non-trivial, since the uniform measure μ_0 restricted to $\mathcal{A}/\mathcal{G}$ is the null measure. This fact has put in evidence the important role of the generalized connections in the loop quantization.

Finally, let's remember the general algorithm of canonical quantization:

1. Pick a Poisson algebra of classical quantities;

2. Represent these quantities as quantum operators acting on a Hilbert space of quantum states;

3. Implement any constraint of the theory as a quantum operator equation and solve it to get the physical states;

4. Construct an inner product on physical states;

5. Develop a semiclassical approximation and compute expectation values of physical quantities.

When the theory we want to quantize is gravity the Poisson algebra is generated by the pairs (A^i_a, E^b_j), where A is a connection and E is the canonical moment associated to it.

The classical configuration space is taken to be $\mathcal{A}/\mathcal{G}$ and it is densely embedded in the quantum configuration space $\overline{\mathcal{A}/\mathcal{G}}$.

$\overline{\mathcal{A}/\mathcal{G}}$ carries the gauge and diffeomorphism invariant uniform measure μ_0 that enables to construct the Hilbert space of kinematical states $L^2(\overline{\mathcal{A}/\mathcal{G}}, \mu_0)$.

The gauge-invariant information of the connection is contained in the Wilson functions, these are promoted to multiplication operators on the carrier space $L^2(\overline{\mathcal{A}/\mathcal{G}}, \mu_0)$:

$$T_\alpha \rightsquigarrow \hat{T}_\alpha(\psi) := \tilde{T}_\alpha(\bar{A})\psi(\bar{A})$$

where $\psi \in L^2(\overline{\mathcal{A}/\mathcal{G}}, \mu)$ and T_α has to be interpreted as its Gelfand transform, which lies in $\mathcal{C}(\overline{\mathcal{A}/\mathcal{G}}) \subset L^2(\overline{\mathcal{A}/\mathcal{G}}, \mu_0)$.

As usual in canonical quantization, the conjugate momenta are promoted to a derivative operator-valued distribution:

$$E^a_i(x) \rightsquigarrow -i\frac{\delta}{\delta A^i_a}(x).$$

The Wilson loop operators $\hat{T}_\alpha$ are bounded self-adjoint operators on $L^2(\overline{\mathcal{A}/\mathcal{G}}, \mu)$ for every μ. It is the requirement that the momentum operators must be self-adjoint that restricts the measure, i.e. they satisfy self-adjointness if and only if $\mu = \mu_0$. A similar situation occurs already in non-relativistic quantum mechanics: while the position operator $\hat{Q}$ is self-adjoint on $L^2(R, f(x)dx)$ for any (regular) function $f : \mathbb{R} \to \mathbb{R}$, the momentum operator $P = -i\frac{d}{dx}$ is self adjoint if and only if f is a constant. Thus, it is the self-adjointness of the explicitly defined momentum operators that singles out the Lebesgue measure dx.

As said before, the assumption of $\mathcal{A}/\mathcal{G}$ as classical configuration space solve already at the classical level the Gauss constraint generated by the invariance under gauge transformations.

Hence we can say that the kinematical (gauge-invariant) configuration space of loop quantum gravity is selected only by two essential inputs:

1. it must carry a representation of the C^*-algebra of classical gauge-invariant configuration observables, i.e. the holonomy C^*-algebra. This leads to $L^2(\overline{\mathcal{A}/\mathcal{G}}, \mu)$.

2. it must be such that the momentum operator is self-adjoint, and this singles out the measure μ to be μ_0.

We stress that both these assumptions seem natural from a mathematical physics perspective.

The diffeomorphism constraint instead is imposed at the quantum level by selecting a suitable subspace of the kinematical state space $L^2(\overline{\mathcal{A}/\mathcal{G}}, \mu_0)$ given by the states satisfying this constraint.

In the canonical quantization one operates the splitting of the space-time in space+time, hence there are two kind of constraints generated by the invariance under diffeomorphism: the constraint depending on the spatial part and that depending on the temporal evolution, called **Hamiltonian constraint**.

There is an important instrument to construct spatial diffeomorphism invariant states called **loop transform**, this is a linear operator that enables to pass from the quantum representation described above, called **connection representation** (because the states are functions of generalized connections), to the so-called **loop representation**, where the states are functions of loops.

The importance of the loop representation is that this representation carries topological invariants, exhibited by the loop transform and called **generalized knot-invariants**, which naturally satisfy the diffeomorphism constraint. To see this notice that the diffeomorphism constraint maps a Wilson function T_α to a Wilson function $T_{\phi\circ\alpha}$ relative to the loop α deformed by a spatial diffeomorphism ϕ, hence it is immediate to understand that knot invariants should play an important role in solving the diffeomorphism constraint.

Rovelli and Smolin were the first ones to construct (formal) solution to the diffeomorphism constraint in [81] starting from the next consideration: it is well known that, in ordinary quantum mechanics, the Fourier transform enables to pass from the **position representation**, in which the states are functions of the generalized coordinates q^i, to the **momentum representation**, in which the states are functions of the momentum coordinates p_j. The Fourier transform is a unitary operator from $L^2(\mathbb{R}^3)$ into itself, thus the momentum and the position representation are two physically equivalent description of the quantum 3-D world since unitary operators preserve the scalar brackets and then the expectation values of the observables. The usefulness of the unitary correspondence between 'position states' and 'momentum states' induced by the Fourier transform is due to the fact that in many situations of physical and mathematical interest the equations of quantum mechanics are much more easily solved in the momentum representation.

Rovelli and Smolin proposed in the already quoted article a formal transform, called **loop transform** and deeply related to the Fourier transform, to pass from the 'connection representation', in which the states are the normalized vectors $|\psi> \in L^2(\overline{\mathcal{A}/\mathcal{G}}, \mu_0)$, to a 'loop representation', in which the states are functions of loops.

The advantage is that it is easier to find out diffeomorphism invariant functions of loops instead of diffeomorphism invariant functions of connections.

The proposal of Rovelli and Smolin was only formal in 1990, but after the construction of the uniform measure on $\overline{\mathcal{A}/\mathcal{G}}$ the expression of the loop transform can be written rigorously as

$$L^2(\overline{\mathcal{A}/\mathcal{G}}, \mu_0) \ni \psi \rightsquigarrow \ell_\psi(\alpha) := \int_{\overline{\mathcal{A}/\mathcal{G}}} \overline{T_\alpha(\bar{A})}\psi(\bar{A})d\mu_0(\bar{A})$$

well defined after one identifies the Wilson function T_α with its Gelfand transform, which belongs to $\mathcal{C}(\overline{\mathcal{A}/\mathcal{G}}) \subset L^2(\overline{\mathcal{A}/\mathcal{G}}, \mu_0)$.

As a consequence of the diffeomorphism invariance of μ_0, if ϕ is an orientation preserving diffeomorphism of M, then $\ell_\psi(\alpha) = \ell_\psi(\phi \circ \alpha)$, i.e. ℓ_ψ assumes the same value on every loop obtained by a fixed loop through a diffeomorphic deformation. Thus the states ℓ_ψ satisfy the diffeomorphism invariance in a natural manner.

While the (spatial) diffeomorphism constraint can be controlled into the arena of loop representation of quantum gravity, the solution to the Hamiltonian constraint remains the main open problem of canonical quantum gravity.

For this reason the researches in quantum gravity, starting from the late nineties, has switched to the covariant formulation, where there is no splitting of spacetime into space+time and the Hamiltonian constraint doesn't appear.

The covariant formulation of quantum gravity is contemplated in the so-called 'spin foam models'. We will describe these models in the last two chapters. In the next chapter instead we present the spin network states, an orthonormal basis of the kinematical space $L^2(\overline{\mathcal{A}/\mathcal{G}}, \mu_0)$ which is very useful to construct important physical observables such as the area and volume operators of slices of space at 'frozen' time.

4.14.1 Inductive construction of the loop transform

In [2] was shown how to derive the loop transform from an inductive limit of Fourier transforms on tori for Abelian gauge theories.

Here we generalize the inductive construction to the more general case of non-Abelian compact gauge theories, i.e. when the group G is fixed to be of the type $U(N)$ or $SU(N)$, $N > 1$.

Let's fix the notation:

- the index set J is the set of the subgroups L of $L_\star(M)$ generated by a finite independent family of loops. By $L \leq L'$ we mean that L is a subgroup of L'; J is a directed set with respect to this order;

- to every $L \in J$ we associate $Hom(L,G)/Ad_G$, that we write $\overline{\mathcal{A}/\mathcal{G}}_L$ for simplicity;
- if $L \leq L'$ we define the projection $\pi_{LL'} : \overline{\mathcal{A}/\mathcal{G}}_{L'} \to \overline{\mathcal{A}/\mathcal{G}}_L$ simply by taking the restrictions of the homomorphisms $H \in Hom(L',G)$ to the subgroup L in each Ad_G-equivalence class.

The spectrum $\overline{\mathcal{A}/\mathcal{G}}$ of $Hol(M,G)$ is the projective limit of the family $\{\overline{\mathcal{A}/\mathcal{G}}_L\}_{L\in J}$ so that, explicitly:

$$\overline{\mathcal{A}/\mathcal{G}} = \varprojlim_{L\in J} \overline{\mathcal{A}/\mathcal{G}}_L \equiv \varprojlim_{L\in J} Hom(L,G)/Ad_G \ .$$

For a given independent family of loops $(\alpha_1, ..., \alpha_n)$ the evaluation map $ev_{(\alpha_1,...,\alpha_n)} : Hom(L,G) \to G^n$, defined by

$$ev_{(\alpha_1,...,\alpha_n)}(H) = (H(\alpha_1), ..., H(\alpha_n))$$

is an isomorphism and factorizes to a homeomorphism from $Hom(L,G)/Ad_G$ to G^n/Ad_G.

We want to construct the loop transform as an inductive limit of Fourier-Plancherel transforms from $L^2(Hom(L,G)/Ad_G)$ onto $L^2_{Ad}\left(\widehat{Hom(L,G)}\right)$[2]. To simplify the notation we write the former spaces as $L^2(\overline{\mathcal{A}/\mathcal{G}}_L)$ and the latter spaces as $L^2\left(\widehat{\overline{\mathcal{A}/\mathcal{G}}_L}\right)$, this is only a symbolic notation because $\overline{\mathcal{A}/\mathcal{G}}_L$ is not a group and so $\widehat{\overline{\mathcal{A}/\mathcal{G}}_L}$ doesn't refer to its dual object.

The scheme of the inductive construction is expressed in the following diagram:

$$\begin{array}{ccc}
\vdots & \vdots & \vdots \\
L^2(\overline{\mathcal{A}/\mathcal{G}}_L) & \xrightarrow{F_L} & L^2\left(\widehat{\overline{\mathcal{A}/\mathcal{G}}_L}\right) \\
\downarrow i_{L'L} & \vdots & \downarrow j_{L'L} \\
L^2(\overline{\mathcal{A}/\mathcal{G}}_{L'}) & \xrightarrow{F_{L'}} & L^2\left(\widehat{\overline{\mathcal{A}/\mathcal{G}}_{L'}}\right) \\
\vdots & \vdots & \vdots \\
\downarrow & \downarrow & \downarrow \\
\varinjlim_L L^2(\overline{\mathcal{A}/\mathcal{G}}_L) & \xrightarrow{\mathcal{L}} & \varinjlim_L L^2\left(\widehat{\overline{\mathcal{A}/\mathcal{G}}_L}\right) .
\end{array}$$

[2]For every group G, $L^2_{Ad}(\hat{\mathsf{G}})$ is the Hilbert subspace of $L^2(\hat{\mathsf{G}})$ generated by the Fourier-Plancherel transforms of the characters χ_ρ, i.e. $L^2_{Ad}(\hat{\mathsf{G}}) := \overline{\text{span}\{\hat{F}\chi_\rho,\ \rho \in \hat{\mathsf{G}}\}}$. The restriction of the Fourier-Plancherel transform to $L^2(\mathsf{G}/Ad_{\mathsf{G}})$ is a unitary operator $F : L^2(\mathsf{G}/Ad_{\mathsf{G}}) \to L^2_{Ad}(\hat{\mathsf{G}})$.

To make the diagram concrete we have to construct the inclusions $i_{L'L}$ and $j_{L'L}$.

If $L \le L'$, then we put, for every $\psi \in L^2(\overline{\mathcal{A}/\mathcal{G}}_L)$:

$$i_{L'L}\psi(H') := \psi(\pi_{LL'}(H')) \qquad H' \in \overline{\mathcal{A}/\mathcal{G}}_{L'}$$

this inclusions are obviously linear and satisfy the consistency conditions. They are also isometric maps, in fact if L and L' are the free groups generated by the independent families $\{\alpha_1, \ldots, \alpha_n\}$ and $\{\beta_1, \ldots, \beta_{n'}\}$, respectively, then every loop of L can be decomposed in a suitable product of loop of L' in this way:

$$\begin{cases} \alpha_1 = {\beta_1}^{k_{1,1}} \ldots {\beta_{n'}}^{k_{n',1}} \\ \vdots \\ \alpha_n = {\beta_1}^{k_{1,n}} \ldots {\beta_{n'}}^{k_{n',n}} \end{cases}$$

with $k_{r,s} \in \mathbb{Z}$, $r = 1, \ldots, n'$ and $s = 1, \ldots, n$.

So, if

$$ev_{(\beta_1,\ldots,\beta_{n'})}(H') = (g_1, \ldots, g_{n'})$$

then

$$ev_{(\alpha_1,\ldots,\alpha_n)}(\pi_{LL'}(H')) = (g_1^{k_{1,1}} \cdots g_{n'}^{k_{n',1}}, \ldots, g_1^{k_{1,n}} \cdots g_{n'}^{k_{n',n}}) \equiv (\tilde{g}_1, \ldots, \tilde{g}_{n'})\,.$$

By composing the evaluation maps with the inclusions $i_{L'L}$ one gets the inclusions $i_{n'n} : L^2(G^n/Ad_G) \to L^2(G^{n'}/Ad_G)$, where $i_{n'n}(\psi) := \psi' \in L^2(G^{n'}/Ad_G)$ acts just like ψ on the first $\tilde{g}_1, \ldots, \tilde{g}_n$ arguments and it is constant on the other $n' - n$ ones. Moreover, since we are dealing with normalized and the bi-invariant Haar measures, we have:

$$\begin{aligned} {\|\psi'\|_{n'}}^2 &= \frac{1}{vol(G)^{n'}} \int_G |\psi'(\tilde{g}_1, \ldots, \tilde{g}_{n'})|^2 d\tilde{g}_1 \cdots d\tilde{g}_{n'} \\ &= \frac{1}{vol(G)^{n'}} \int_0^{2\pi} |\psi(\tilde{g}_1, \ldots, \tilde{g}_n)|^2 d\tilde{g}_1 \cdots d\tilde{g}_n \int_G d\tilde{g}_{n+1} \cdots \int_G d\tilde{g}_{n'} \\ &= \frac{vol(G)^{n'-n}}{(vol(G))^{n'}} \int_G |\psi(\tilde{g}_1, \ldots, \tilde{g}_n)|^2 d\tilde{g}_1 \cdots d\tilde{g}_n \\ &= \frac{1}{(vol(G)^n} \int_G |\psi(\tilde{g}_1, \ldots, \tilde{g}_n)|^2 d\tilde{g}_1 \cdots d\tilde{g}_n \\ &= {\|\psi\|_n}^2. \end{aligned}$$

This computation shows that the inclusions $i_{n'n}$, and hence also the inclusions $i_{L'L}$, are isometric maps.

The inclusions $j_{L'L}$ are defined by means of the following commutative diagram:

$$\begin{array}{ccc} L^2(\overline{\mathcal{A}/\mathcal{G}}_{L'}) & \stackrel{F_{L'}}{\longrightarrow} & L^2\left(\widehat{\overline{\mathcal{A}/\mathcal{G}}_{L'}}\right) \\ \uparrow i_{L'L} & & \uparrow j_{L'L} \\ L^2(\overline{\mathcal{A}/\mathcal{G}}_{L}) & \stackrel{F_{L}}{\longrightarrow} & L^2\left(\widehat{\overline{\mathcal{A}/\mathcal{G}}_{L}}\right) \end{array}$$

They are obviously isometric maps since they are compositions of isometries, furthermore, with such a definition, the consistency condition holds both for the inclusions $j_{L'L}$ and the Fourier-Plancherel transforms F_L.

Now that we have constructed the inductive families of Hilbert spaces and isometric transforms between them we have to investigate their inductive limits.

We will make use of the following important property of the inductive limits, which we specialize to the case of interest for us:

Theorem 4.14.1 (Universality of the inductive limit) *Let $(\mathcal{H}_\mu, i_{\nu\mu}, J)$ be an inductive family of Hilbert spaces with inductive limit $\mathcal{H}$. If there exists an Hilbert space $\tilde{\mathcal{H}}$ and isometric linear maps $i_\mu : \mathcal{H}_\mu \to \tilde{\mathcal{H}}$, $\mu \in J$, such that $i_\mu = i_\nu \circ i_{\nu\mu}$ for every $\mu \leq \nu$ then there is an unique isometric linear map i such that $i_\mu = i \circ i_\mu$ for every μ. Furthermore, if the ranges of the maps i_μ span a dense set in $\mathcal{H}$, then $\tilde{\mathcal{H}} = \mathcal{H}$.*

Theorem 4.14.2 *The following assertions hold.*

1. *The inductive limit of $\{L^2(\overline{\mathcal{A}/\mathcal{G}}_L), i_{L'L}\}_{L\in J}$ is $L^2(\overline{\mathcal{A}/\mathcal{G}}, \mu_0)$;*
2. *The inductive limit of $\{F_L\}_{L\in J}$ is a unitary map $\mathcal{L}$ whose image is the inductive limit of $\left\{L^2\left(\widehat{\overline{\mathcal{A}/\mathcal{G}}_L}\right), i_{L'L}\right\}_{L\in J}$.*

Proof. We use the universality property stated above. Define the inclusion $i_L : L^2(\overline{\mathcal{A}/\mathcal{G}}_L) \to L^2(\overline{\mathcal{A}/\mathcal{G}}, \mu_0)$ as the pull-back of the projection $\pi_L : \overline{\mathcal{A}/\mathcal{G}} \to \overline{\mathcal{A}/\mathcal{G}}_L$:

$$(i_L \psi_L)(H_L) := \psi_L(\pi_L(H_L)) \qquad \psi_L \in L^2(\overline{\mathcal{A}/\mathcal{G}}_L)$$

for every L. The proof that i_L is an isometry is easy if we make use of the cylindrical nature of the uniform measure μ_0: for every $\psi_L \in L^2(\overline{\mathcal{A}/\mathcal{G}}_L)$, the function $\psi := i_L \psi_L$ is a cylindrical function of $L^2(\overline{\mathcal{A}/\mathcal{G}}, \mu_0)$ w.r.t. the index L, so that:

$$\|\psi\|^2 = \int_{\overline{\mathcal{A}/\mathcal{G}}} |\ \psi\ |^2\, d\mu_0 = \int_{\overline{\mathcal{A}/\mathcal{G}}_L} |\ \psi_L\ |^2\, d\mu_L = \|\psi_L\|^2\ .$$

Suppose now that a function $\psi \in L^2(\overline{\mathcal{A}/\mathcal{G}}, \mu_0)$ is orthogonal to all the pull-backs $i_L\,\psi_L$, $\psi_L \in L^2(\overline{\mathcal{A}/\mathcal{G}}_L)$, then ψ is orthogonal to all the cylindrical functions of $L^2(\overline{\mathcal{A}/\mathcal{G}}, \mu_0)$, but they form a dense subset of $L^2(\overline{\mathcal{A}/\mathcal{G}}, \mu_0)$ so the orthogonality of ψ extends to all the functions of $L^2(\overline{\mathcal{A}/\mathcal{G}}, \mu_0)$ and then ψ must be the zero function.

The inclusions $j_L : L^2\left(\widehat{\overline{\mathcal{A}/\mathcal{G}}_L}\right) \to L^2\left(\widehat{\overline{\mathcal{A}/\mathcal{G}}}\right)$ are defined by means of the following diagram

$$\begin{array}{ccc} L^2(\overline{\mathcal{A}/\mathcal{G}}) & \xrightarrow{\mathcal{L}} & L^2\left(\widehat{\overline{\mathcal{A}/\mathcal{G}}}\right) \\ {\scriptstyle i_L}\Big\uparrow & & \Big\uparrow{\scriptstyle j_L} \\ L^2(\overline{\mathcal{A}/\mathcal{G}}_L) & \xrightarrow[F_L]{} & L^2\left(\widehat{\overline{\mathcal{A}/\mathcal{G}}_L}\right) \end{array}$$

being $\mathcal{L} := \varinjlim_{L \in J} F_L$. Repeating the considerations above on the inclusions j_L and applying the consequences of the Peter-Weyl theorem on the maps F_L, the remaining part of the theorem is fully proven. □

Def. 4.14.1 *We call the unitary map $\mathcal{L}$ defined in the previous theorem the 'inductive loop transform'.*

In [2] it was proven that the image of the inductive loop transform for gauge theories with symmetry group $U(1)$, as the electromagnetic theory, is the Hilbert space $L^2(H_\star(M))$ w.r.t the discrete measure, where $H_\star(M)$ is the hoop group, i.e. the loop group $L_\star(M)$ modulo its commutator subgroup. Unfortunately, in the compact non-Abelian case such a characterization is not available due to the fact that the Ad-equivalence is not trivial.

Finally notice that the inductive loop transform constructed here is unitarily equivalent to the spin network transform introduced by Thiemann in [90], this follows again from the Peter-Weyl theorem by remembering that the spin network states constitute an orthonormal basis for $L^2(\overline{\mathcal{A}/\mathcal{G}})$ (see next chapter). Therefore, the loop representation defined by the inductive loop transform and the spin network representation that will be exposed in the following chapter are physically indistinguishable.

Chapter 5

Spin network states, area and volume operators

In this chapter we present the very elegant construction of an orthonormal basis of $L^2(\overline{\mathcal{A}/\mathcal{G}}, \mu_0)$ due to Baez [16], [17]. This basis is given by the so-called spin network states, introduced for the first time in loop quantum gravity by Rovelli and Smolin in [83], inspired by an old idea of Penrose [74].

The spin network states solve the problem of overcompleteness of the Wilson functions and, at the same time, enables to write down the explicit form of the volume and area operators, the two fundamental observables of loop quantum gravity. We will give a brief account of this fact in the second section, after the rigorous introduction of the spin network states.

5.1 Construction of the spin network states

Fix a principal bundle $P(M, G)$ with $M \simeq \mathbb{R} \times \Sigma$, where Σ is a spacelike hypersurface and $G = SU(2)$.

Then take a graph Γ in Σ and consider the Hilbert space $L^2(\overline{\mathcal{A}}_\Gamma)$. Since $\overline{\mathcal{A}}_\Gamma \simeq G^{E_\Gamma}$, this space can be written as:

$$L^2(\overline{\mathcal{A}}_\Gamma) \simeq \bigotimes_{e \in E_\Gamma} L^2(G).$$

Now one can apply the Peter-Weyl theorem to decompose $L^2(G)$, obtaining:

$$L^2(\overline{\mathcal{A}}_\Gamma) \simeq \bigotimes_{e \in E_\Gamma} \bigoplus_{\lambda \in \hat{G}} \mathfrak{M}^\lambda \otimes \overline{\mathfrak{M}^\lambda}.$$

The decomposition above can be re-written in a more convenient form: let $\hat{G}^{E_\Gamma}$ be the set of all the labelling of the edges of Γ with irreducible unitary representations of G, then, by using the distributivity of the tensor product with respect to the direct sum, it follows that $L^2(\overline{\mathcal{A}}_\Gamma)$ is isomorphic to:

$$L^2(\overline{\mathcal{A}}_\Gamma) \simeq \bigoplus_{\lambda \in \hat{G}^{E_\Gamma}} \bigotimes_{e \in E_\Gamma} \mathfrak{M}^\lambda \otimes \overline{\mathfrak{M}^\lambda}.$$

Now, if $S(v)$ is the set of edges of Γ having *source* in the vertex v and $T(v)$ is the set of edges of Γ having *target* in the vertex v, then:

$$L^2(\overline{\mathcal{A}}_\Gamma) \simeq \bigoplus_{\lambda \in \hat{G}^{E_\Gamma}} \bigotimes_{v \in V_\Gamma} \left(\bigotimes_{e \in S(v)} \mathfrak{M}^\lambda \otimes \bigotimes_{e \in T(v)} \overline{\mathfrak{M}^\lambda} \right).$$

The convenience of this last decomposition is that, being all the representations unitary, all the spaces $\overline{\mathfrak{M}^\lambda}$ are ([28]) isomorphic with $(\mathfrak{M}^\lambda)^*$, hence the last decomposition can be written as:

$$L^2(\overline{\mathcal{A}}_\Gamma) \simeq \bigoplus_{\lambda \in \hat{G}^{E_\Gamma}} \bigotimes_{v \in V_\Gamma} Hom \left(\bigotimes_{e \in T(v)} \mathfrak{M}^\lambda, \bigotimes_{e \in S(v)} \mathfrak{M}^\lambda \right).$$

Now remember from chapter 2 that, given two representations $\rho : G \to Aut(V)$ and $\rho' : G \to Aut(V')$ of the group G, one can always construct a representation of G supported on $Hom(V, V')$ in this way:

$$\begin{array}{rccl} \eta : & G & \longrightarrow & Aut(Hom(V, V')) \\ & g & \mapsto & \eta_g, \end{array}$$

$\eta_g(A) := \rho'_g \circ A \circ \rho_{g^{-1}}$, for every $A \in Hom(V, V')$.

In the situation under analysis the representation ρ can be specialized to be $\bigotimes_{e \in T(v)} L|_{\mathfrak{M}^\lambda}$ and ρ' to be $\bigotimes_{e \in S(v)} R|_{\mathfrak{M}^\lambda}$, where L and R are the regular left and right representations of G, respectively.

Finally observe that for every *fixed* vertex v of the graph Γ, the action of $\overline{\mathcal{G}}_\Gamma$ on $Hom\left(\bigotimes_{e \in T(v)} \mathfrak{M}^\lambda, \bigotimes_{e \in S(v)} \mathfrak{M}^\lambda\right)$ identifies with the action η (with the specializations above) of the group G on the same space, hence, as proved in chapter 2, its gauge-invariant subspace is given precisely by the linear span of the intertwining operators between the indicated representations, thus:

$$L^2(\overline{\mathcal{A}/\mathcal{G}_\Gamma}) \simeq \bigoplus_{\lambda \in \hat{G}^{E_\Gamma}} \bigotimes_{v \in V_\Gamma} Int \left(\bigotimes_{e \in T(v)} \mathfrak{M}^\lambda, \bigotimes_{e \in S(v)} \mathfrak{M}^\lambda \right).$$

Now the definitions of spin network and spin network state can be stated.

Def. 5.1.1 *A* **spin network** *is a triple* $(\Gamma, \vec{\rho}, \vec{I})$ *consisting of:*

- *a graph* Γ *in* M*;*
- *a labelling* $\vec{\rho}$ *of each edge* e *of the graph* Γ *with a nontrivial irreducible representation* ρ_e *of* G*;*
- *a labelling* $\vec{I}$ *of each vertex of* Γ *with an intertwining operator* I_v *from the tensor product of the representations* ρ_e *associated to the edges* e *incoming in* v *to the tensor product of the representations* ρ_e *associated to the edges* e *outgoing from* v*.*

Given a spin network $(\Gamma, \vec{\rho}, \vec{I})$*, the* **spin network state** *based on it is the function on* $\mathcal{A}/\mathcal{G}$ *constructed as follows:*

$$\begin{array}{rccl} \psi_{(\Gamma,\vec{\rho},\vec{I})}: & \mathcal{A}/\mathcal{G} & \longrightarrow & \mathbb{C} \\ & [A] & \mapsto & \psi_{(\Gamma,\vec{\rho},\vec{I})}(A) := [\bigotimes_e \rho_e(H_A(e))] \cdot [\bigotimes_v I_v] \end{array}$$

where $A \in [A]$ *and the dot '$\cdot$' stands for contracting, at each vertex* v *of the graph* Γ*, the upper indices of the matrixes corresponding to the incoming edges in* v*, the lower indexes of the matrixes assigned to the outgoing edges in* v*, and the corresponding upper and lower indices of the intertwiners* I_v*.*

A spin network state $\psi_{(\Gamma,\vec{\rho},\vec{I})}$ is a gauge-invariant cylindrical function of connections since it depends only on a finite number of holonomies and, obviously, the Wilson functions are particular cases of spin network states.

Being the space of cylindrical functions an Abelian C^*-algebra with unit, we know that, thanks to the Gelfand isomorphism, we can extend in a unique way the spin network states to cylindrical functions on $\overline{\mathcal{A}/\mathcal{G}}$.

Since $L^2(\overline{\mathcal{A}/\mathcal{G}_\Gamma})$ is the completion of $Cyl_\Gamma(\overline{\mathcal{A}/\mathcal{G}})$ and since

$$\overline{\bigcup_\Gamma L^2(\overline{\mathcal{A}/\mathcal{G}_\Gamma})} = L^2(\overline{\mathcal{A}/\mathcal{G}})$$

the spin network states, as Γ varies, span $L^2(\overline{\mathcal{A}/\mathcal{G}})$.

Moreover the previous decomposition of $L^2(\overline{\mathcal{A}/\mathcal{G}})$ shows that they are orthonormal by construction and Baez shown in [16] that they satisfy the consistency conditions to form and inductive family.

All these considerations lead to the following final result.

Theorem 5.1.1 *The spin network states* $\{\psi_{(\Gamma,\vec{\rho},\vec{I})}\}_\Gamma$ *form a complete orthonormal system for* $L^2(\overline{\mathcal{A}/\mathcal{G}})$*.*

Notice that, in standard quantum field theory, a spin-network-like computation gives rise to the celebrated Feynman diagrams. In fact we compute transition amplitudes between quantum states as sums or integrals over graphs with edges labelled by irreducible unitary representations of the relevant symmetry group. Typically this group is the product of the Poincaré group and some internal symmetry group, so the edges are labelled by momenta, spins and certain internal quantum numbers. To compute the transition amplitude from one basis state to another, we sum over graphs going from one set of points labelled by representations (and vectors lying in these representations) to some other such set.

The contribution of any graph to the amplitude is given by a product of amplitudes associated to its vertices and edges. Each vertex amplitude depends only on the representations labelling the incident edges, while each edge amplitude, or propagator, depends only on the label of the edge itself.

5.1.1 An example of spin network: the 'theta'

We want to give an example of calculation of a spin network state considering the so-called 'theta' spin network (the name is due to the particular form of the graph underlying the spin network, as can be seen in figure below).

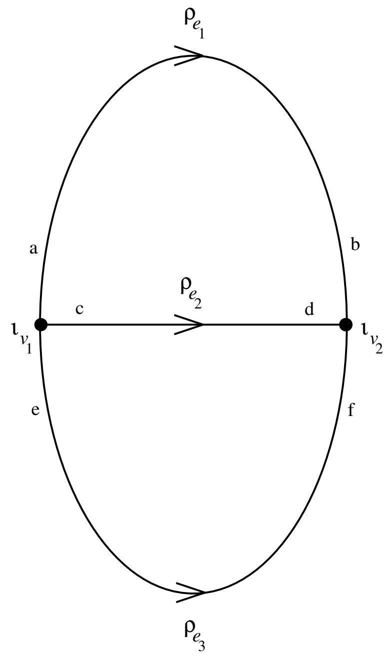


Figure 5.1: An example of spin network: the Θ.

In this spin spin network there are three edges e_1, e_2, e_3 and two vertices v_1, v_2.

We draw arrows on the edges to indicate their orientation. For any connection $A \in \mathcal{A}$ the spin network state associated to the spin network above is given by:

$$\psi_{(\Theta,\vec{\rho},\vec{\iota})}(A) = \rho_{e_1}(H_A(e_1))^a_b\, \rho_{e_2}(H_A(e_2))^c_d\, \rho_{e_3}(H_A(e_3))^e_f\, (\iota_{v_1})_{ace}\, (\iota_{v_2})^{bdf}.$$

In other words, we take the holonomy along each edge of the $\Theta-$graph, think of it as a group element, and put it into the representation labelling that edge.

Picking a basis for this representation we think of the result as a matrix with one superscript and one subscript. We use the little letter near the beginning of the edge for the superscript and the little letter near the end of the edge for the subscript.

In addition, we write the intertwining operator for each vertex as a tensor. This tensor has one superscript for each edge incoming to the vertex and one subscript for each edge outgoing from the vertex. Note that this recipe ensures that each letter appears once as a superscript and once as a subscript!

Finally, using the Einstein summation convention we sum over all repeated indices and get a number, which of course depends on the connection A. This is $\psi_{(\Theta,\vec{\rho},\vec{\iota})}(A)$.

5.2 Area and volume operators and their spectra

Since the spin network states are an orthonormal basis for $L^2(\overline{\mathcal{A}/\mathcal{G}}, \mu_0)$, we can use them to define the action of the operators on this Hilbert space.

In particular we are interested in the operators that correspond to area and volume, because they will give us the picture of the space at the Planck scale!

Here we only want to present a sketch of the quantization procedure of the area of a surface S and the volume V of a region R in Σ. At a classical level, as we have seen in chapter 1, area and volume are given by:

$$A_S = \int_S \sqrt{E^a_i E^b_i n_a n_b}\, dx^1 \wedge dx^2;$$

$$V_R := \int_R \sqrt{det(E^a_i)}\, dx^1 \wedge dx^2 \wedge dx^3;$$

where n_A is the normal to the surface S.

The first calculation of the area and volume operators in loop quantum gravity was done by Rovelli and Smolin in [82]. A more rigorous construction, that uses a different quantization scheme, was given by Ashtekar and Lewandowski in [10], [11]. Here we follow the simplified derivation of the area operator that can be found in [84].

For simplicity assume that S is defined by $x^3 = 0$ and the coordinates x^1 and x^2 parameterize it, then the area becomes

$$A_S = \int_S \sqrt{E_i^3 E^{3i}}\, dx^1 \wedge dx^2.$$

The naive substitution of the functional derivative $-i\frac{\delta}{\delta A_a^i}$ to the field E_i^a gives an operator-valued distribution, so that, in order to have a well defined operator, we have to 'smear' the distribution, i.e. to regularize it by an opportune integration:

$$\hat{E}^i(S) = -i \int_S dx^1 \wedge dx^2\, \varepsilon_{abc} \frac{\partial y^a}{\partial x^1} \frac{\partial y^b}{\partial x^2} \frac{\delta}{\delta A_c^i(y(x))}.$$

The operator $\hat{E}^i(S)$ is well defined on the cylindrical functions. To see how this may happen, consider its action on the spin network state $\psi_\Gamma \equiv \psi_{\Gamma,\vec{\rho},\vec{I}}$. We can think at the intersection points between the spin network $\Gamma \equiv (\Gamma, \vec{\rho}, \vec{I})$ and the surface S as 'punctures'. To begin with, let us consider the simplest case in which the surface S and the spin network Γ intersect on single puncture p, where p lies on (the interior of) the edge

$$\gamma : \mathbb{R} \to S, \quad t \mapsto x^a(t)$$

labelled by the spin j. A standard result to compute the action of the functional derivative on the holonomies can be obtained by taking the first variation of the differential equation that defines the holonomy of the connection A along γ, i.e. $H_A(\gamma_t) + A(\dot{\gamma}_t)H_A(\gamma_t) = 0$ (see chapter 4). The result is that the derivative of the matrix $H_A(\gamma)$ in the spin-j representation is

$$\frac{\delta}{\delta A_a^i(x)} j(H_A(\gamma)) = \int_\gamma dt \frac{dx^a(t)}{dt} \delta^3(\gamma_t, x) j(H_A(\gamma_{0\to t})) X_{(j)}^i j(H_A(\gamma_{t\to 1}))$$

where $\gamma_{0\to t}$ and $\gamma_{t\to 1}$ denote the two segments in which the point with coordinate t cuts γ and where $X_{(j)}^i$ are the generators of the spin-j representation of $SU(2)$.

Now, if we isolate $j(H_A(\gamma))$ in the spin network state ψ_Γ, i.e. we write

$$\psi_\Gamma(A) = \psi_{\Gamma-\gamma}^{lm}(A) j(H_A(\gamma))_{lm}$$

then the action of the operator $\hat{E}^i(S)$ on ψ_Γ is

$$\hat{E}^i(S)\,\psi_\Gamma(A) = -i\int_S dx^1\wedge dx^2\,\varepsilon_{abc}\frac{\partial y^a}{\partial x^1}\frac{\partial y^b}{\partial x^2}\psi^{lm}_{\Gamma-\gamma}(A)\frac{\delta}{\delta A^i_c(y(x))}j(H_A(\gamma))_{lm}.$$

Substituting the expression of the functional derivative obtained above we arrive to

$$\hat{E}^i(S)\,\psi_\Gamma(A) = -i\int_S dx^1\wedge dx^2\int_\gamma dt\,\varepsilon_{abc}\frac{\partial y^a}{\partial x^1}\frac{\partial y^b}{\partial x^2}\frac{dx^c(t)}{dt}\delta^3(\gamma_t, y(x))\,\cdot$$

$$\cdot\,\psi^{lm}_{\Gamma-\gamma}(A)j(H_A(\gamma_{0\to t}))X^i_{(j)}j(H_A(\gamma_{t\to 1})).$$

Remarkably, the three partial derivatives combine to produce the Jacobian for the change of integration coordinates from (x^1, x^2, t) to (y^1, y^2, y^3). If this jacobian is non vanishing, we perform the change of integration coordinates and then we can integrate away the delta function, obtaining

$$\hat{E}^i(S)\psi_\Gamma(A) = -i\psi^{lm}_{\Gamma-\gamma}(A)\left(j(H_A(\gamma_{0\to t}))X^i_{(j)}j(H_A(\gamma_{t\to 1}))\right)_{lm}.$$

Thus, **the effect of the operator $\hat{E}^i(S)$ on the spin network state $\psi_\Gamma(A)$ is simply the insertion of the matrix $-\iota X^i_{(j)}$ in the point corresponding to the puncture**. If, on the other hand, the Jacobian vanishes, then the entire integral vanishes. This happens if the tangent to the edge $\frac{dx^a(t)}{dt}$ is tangent to the surface, in particular, for instance, this happens if the edge lies entirely on the surface, in which case the puncture is not an isolated point. Therefore **only isolated punctures contribute to** $\hat{E}^i(S)\psi_\Gamma(A)$.

The key result of the above computation is the analytical expression for the (integer) intersection number $I(S,\gamma)$ between a surface S and an edge γ, i.e.

$$I(S,\gamma) = -i\int_S dx^1\wedge dx^2\int_\gamma dt\,\varepsilon_{abc}\frac{\partial y^a}{\partial x^1}\frac{\partial y^b}{\partial x^2}\frac{dx^c(t)}{dt}\delta^3(\gamma_t, y(x))$$

this integral is independent of the coordinates and yields an integer: the (oriented)[1] number of punctures.

Finally, it is easy to see what happens if the surface S and the spin network Γ intersect in more than one puncture (along the same or different edges): in this case $\hat{E}^i(S)\psi_\Gamma(A)$ is a sum of one term per puncture, each term being given by the insertion of an X-like matrix.

Unfortunately, a bit more care is required for the computation of $\hat{E}^i(S)\psi_\Gamma(A)$ when the punctures are also nodes of s, for a discussion of this feature see, for instance, Appendix B of [84].

[1]The sign is determined by the relative orientation of surface and edge.

Now let's apply the square of the operator $\hat{E}^i(S)$ on a spin network state ψ_Γ such that the edges of γ intersect S only in a puncture. We find:

$$\begin{aligned}
\hat{E}^2(S)\psi_\Gamma(A) &:= \hat{E}^i(S)\hat{E}^i(S)\psi_\Gamma(A) \\
&= -\psi^{lm}_{\Gamma-\gamma}(A)\left(j(H_A(\gamma_{0\to t}))X^i_{(j)}X^i_{(j)}j(H_A(\gamma_{t\to 1}))\right)_{lm} \\
&= \psi^{lm}_{\Gamma-\gamma}(A)\left(j(H_A(\gamma_{0\to t}))j(j+1)j(H_A(\gamma_{t\to 1}))\right)_{lm} \\
&= j(j+1)\psi_\Gamma(A)
\end{aligned}$$

where we have used the fact that $X^i_{(j)}X^i_{(j)}$ is (-1 times) the Casimir operator in the spin-j representation of $SU(2)$ and we have already seen in chapter 2 that this operator is a scalar operator with eigenvalues $j(j+1)$. Thus the area operator $\sqrt{\hat{E}^i(S)\hat{E}^i(S)}$ for a single puncture has this behavior:

$$\sqrt{\hat{E}^i(S)\hat{E}^i(S)}\psi_\Gamma(A) = \sqrt{j(j+1)}\psi_\Gamma(A).$$

It is easy to extend the definition of the area operator to the case of many punctures, in fact one can consider a sequence of increasingly fine partitions of S in n small surfaces such that in everyone of them there is only one puncture. The area operator generalizes to this situation as well as possible, in fact one can quite easily show that the limit operator, denoted with $\mathsf{A}(S)$ acts in this way of the spin network states:

$$\mathsf{A}(S)\psi_\Gamma = \left(\sum_{i\in\Gamma\cap S}\sqrt{j_i(j_i+1)}\right)\psi_\Gamma$$

where the sum is taken over all points p_i where an edge of the spin network Γ intersects the surface S.

Moreover if we re-introduce the correct dimensions and constants G, c and $\hbar$ we see that they combine to give the square of the Planck length ℓ_P, i.e. **the Planck area ℓ_P^2!**

This result is of huge importance: **the area operator $\mathsf{A}(S)$ is diagonalized on the spin network basis and every spin network edge that punctures S contributes with the integer multiple $\sqrt{j(j+1)}\ell_P^2$ of the Planck area to the area of the surface S, this means that the area is a discrete entity which has as fundamental building block the Planck area!**.

This amazing result is considered the most important indication that, at a very microscopical level, the universe is far from being a continuous smooth structure.

As possible visualization of what just described we can draw this picture:

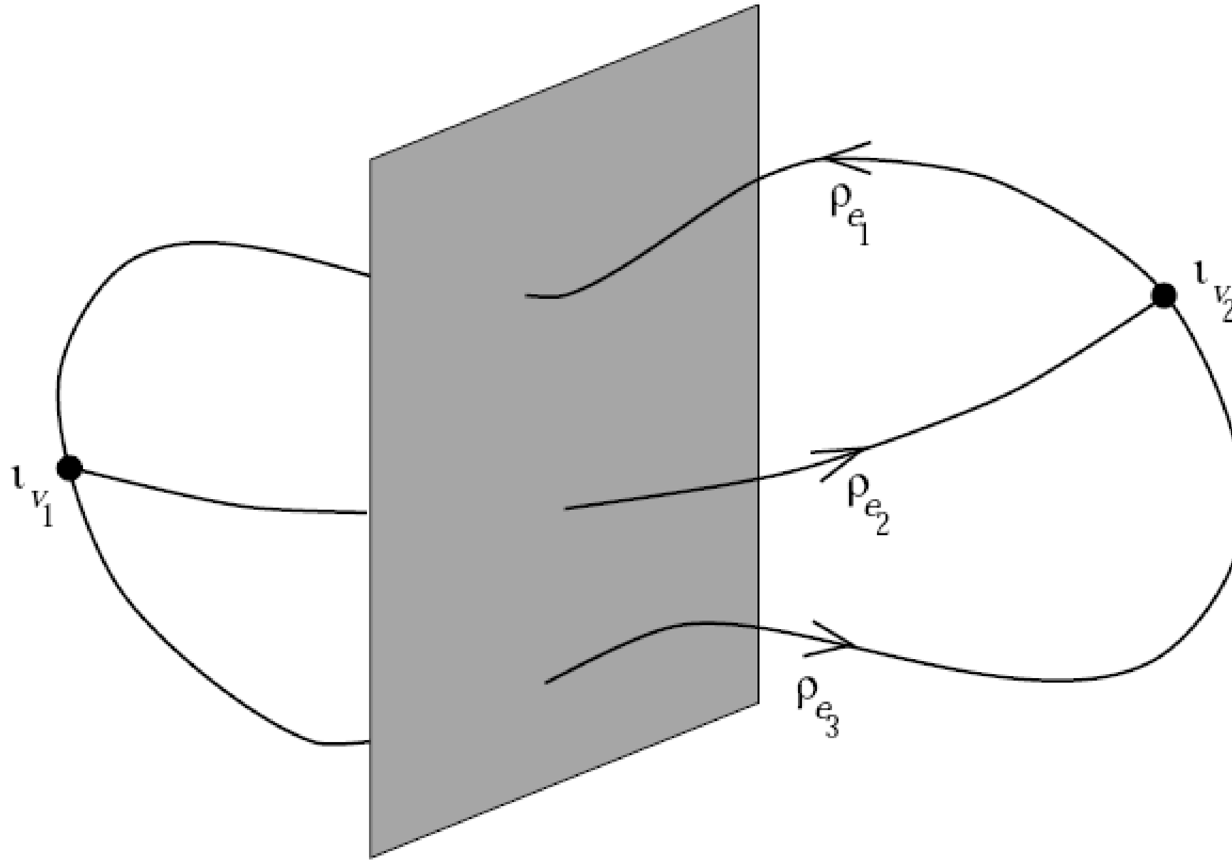


Figure 5.2: Edges of a spin network puncturizing the surface S.

The computation of the area operator has been used by Ashtekar, Baez and Krasnov to compute the entropy of certain black holes [5], the result they found fits the previous result of Hawking and Bekenstein. This could be and indication of the correctness of loop quantum gravity.

We want to conclude the discussion of the area operator by showing what happens in the general case in which the punctures of the edges to the surface S can be also nodes of the spin network Γ.

More precisely that p is an n-valent vertex, we must distinguish the edges that meet in p in three classes, which we denote 'up', 'down' and 'tangential'. The tangential edges are the ones that overlap with S for a finite interval, the others edges are naturally separated into two classes according to the side of S they lie (which we arbitrarily label 'up' and 'down'). We call j^u, j^d and j^t, respectively, the spins of these edges. Then one can show that the area operator generalizes to a self-adjoint operator that diagonalizes on the spin network basis in this way:

$$\mathsf{A}(S)\psi_\Gamma(A) = \frac{\ell_P^2}{2}\left(\sum_{i\in\Gamma\cap S}\sqrt{2j^u(j^u+1)+2j^d(j^d+1)+j^t(j^t+1)}\right)\psi_\Gamma(A).$$

Finally let us say some few words about the volume operator: this operator can be constructed in a rigorous way and it has been proven that it possesses only discrete spectrum as the area operator, but the explicit form of the whole spectrum is still not available. Naively we can say that the vertices here play the role of the edges in the area operator, i.e. they contribute to the computation of the volume of the region in which they lie, with an integer multiple of the Planck volume ℓ_P^3.

The interested reader can find a rigorous discussion of the volume operator in [11].

Chapter 6

Covariant formulation of quantum gravity: spin foam models

This chapter is strongly inspired by the beautiful articles of Baez [18] and [19].

6.1 Introduction: the need of a spin foam models for a covariant quantization of gravity

We have seen that, thanks to the theory of spin networks, loop quantum gravity gives a description of the geometry of *space* at the very microscopical level of the Planck scale, but *not* the geometry of *spacetime*. This happens because loop quantum gravity is based on canonical quantization, in which states describe the geometry of space at a frozen time. The dynamics enters in the theory only in the form of the constraint called the Hamiltonian constraint.

Unfortunately this constraint is still poorly understood and so, we have no idea of what loop quantum gravity might say about the geometry of spacetime.

To remedy this problem, it is natural to try to supplement loop quantum gravity with an appropriate path-integral formalism. In ordinary quantum field theory we calculate path integrals using Feynman diagrams. Copying this idea, in loop quantum gravity we may try to calculate path integrals using spin foams, which are a 2-dimensional analogue of Feynman diagrams.

We have seen that spin networks are graphs with edges labelled by group representations and vertices labelled by intertwining operators. Similarly, a spin foam is a 2-dimensional complex built from vertices, edges and polygonal faces, with the faces labelled by group representations and the edges labelled by intertwining operators. Quoting [18]: *'when the group is $SU(2)$ and three faces meet at each edge, this looks exactly like a bunch of soap suds with all the faces of the bubbles labelled by spins hence the name spin foam'*.

If we take a generic slice of a spin foam, we get a spin network. Thus we can think of a spin foam as describing the geometry of spacetime, and any slice of it as describing the geometry of space at a given time. Ultimately we would like a spin foam model of quantum gravity, in which we compute transition amplitudes between states by summing over spin foams going from one spin network to another:

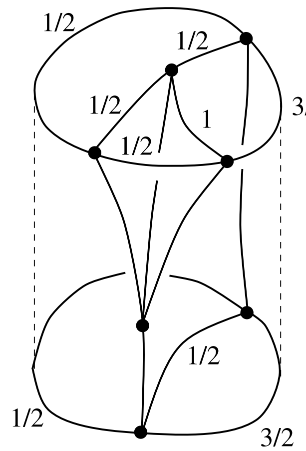


Figure 6.1: An example of spin foam.

At present this goal has been only partially attained. An important fact to stress is that, while canonical loop quantum gravity starts from Ashtekar's formulation of general relativity, the spin foam models start from another description of gravity: general relativity in 4 dimensions can be viewed as a BF theory with extra constraints. Most work on spin foam models of 4-dimensional quantum gravity seeks to exploit this fact.

This is why we start by describing BF theory at the classical level and then we show how it reduces to general relativity with the introduction of suitable constraints. Next we propose how to generally construct a spin foam model and finally we discuss one of the most promising approaches to the spin foam model of quantum gravity: the Barrett-Crane model.

6.2 BF theory and gravity

To set up BF theory, we take as our gauge group any semisimple Lie group G whose Lie algebra $\mathfrak{g}$ is equipped with an invariant non-degenerate bilinear form $< , >$.

We take as our spacetime any n-dimensional oriented smooth manifold M, and choose a principal G-bundle P over M. The basic fields in the theory are then:

- a connection A on P;
- an $ad(P)$-valued $(n-2)$-form E on M.

Remember that $ad(P)$ is the vector bundle associated to P via the adjoint action of G on its Lie algebra. The curvature of A is an $ad(P)$-valued 2-form F on M. If we pick a local trivialization we can think of A as a $\mathfrak{g}$-valued 1-form on M, F as a $\mathfrak{g}$-valued 2-form, and E as a $\mathfrak{g}$-valued $(n-2)$-form.

The Lagrangian density for BF theory is:

$$\mathcal{L} = Tr(E \wedge F)$$

where $Tr(E \wedge F)$ is the n-form constructed by taking the wedge product of the differential form parts of E and F and using the bilinear form $< , >$ to pair their $\mathfrak{g}$-valued parts.

The notation Tr refers to the fact that when G is semisimple the bilinear form is the Killing form $< X, Y >= Tr(ad\, X, ad\, Y)$.

We obtain the field equations by setting the variation of the action to zero:

$$\begin{aligned} 0 &= \delta \int_M \mathcal{L} \\ &= \int_M Tr(\delta E \wedge F + E \wedge \delta F) \\ &= \int_M Tr(\delta E \wedge F + E \wedge d_A \delta A) \\ &= \int_M Tr(\delta E \wedge F + (-1)^{n-1} d_A E \wedge \delta A) \end{aligned}$$

where dA stands for the exterior covariant derivative. In the second step we used the identity $\delta F = d_A \delta A$, while in the final step we did an integration by parts. We see that the variation of the action vanishes for all δE and δA if and only if the following field equations hold:

$$F = 0, \qquad d_A E = 0.$$

These equations suggest that BF theory is a topological field theory! In fact, all solutions of these equations look the same locally, so BF theory describes a world with no local degrees of freedom. To see this, first note that the equation $F = 0$ says the connection A is flat. Indeed, all flat connections are locally the same up to gauge transformations. The equation $d_A E = 0$ is a bit subtler, in fact it is not true that all solutions of this are locally the same up to a gauge transformation in the usual sense, however BF theory has another sort of symmetry. Suppose we define a transformation of the A and E fields by

$$A \mapsto A, \qquad E \mapsto E + d_A \eta$$

for some $ad(P)$-valued $(n-3)$-form η. This transformation leaves the action unchanged, in fact:

$$\begin{aligned} \int_M Tr((E + d_A\eta) \wedge F) &= \int_M Tr(E \wedge F + d_A\eta \wedge F) \\ &= \int_M Tr(E \wedge F + (-1)^n \eta \wedge d_A F) \\ &= \int_M Tr(E \wedge F) \end{aligned}$$

where we used integration by parts and the Bianchi identity $d_A F = 0$. This transformation is a gauge symmetry of BF theory, in the more general sense of the term, meaning that two solutions differing by this transformation should be counted as physically equivalent. Moreover, when A is flat, any E field with $d_A E = 0$ can be written locally as $d_A \eta$ for some η; this is an easy consequence of the fact that locally all closed forms are exact. Thus locally, all solutions of the BF theory field equations are equal modulo gauge transformations and transformations of the above sort.

General relativity in 3 dimensions is a special case of BF theory. To see this, take n = 3, let $G = SO(2,1)$, and let $< \, , \, >$ be minus the Killing form. Suppose first that $E : TM \to ad(P)$ is one-to-one. Then we can use it to define a Lorentzian metric on M as follows: $g(v, w) =< Ev, Ew >$ for any tangent vectors $v, w \in T_x M$. We can also use E to pull back the connection A to a metric-preserving connection Γ on the tangent bundle of M.

The equation $d_A E = 0$ then says precisely that Γ is torsion-free, so that Γ is the Levi-Civita connection on M. Similarly the equation $F = 0$ implies that Γ is flat, thus the metric is flat.

In 3 dimensional spacetime, the vacuum Einstein equations simply say that the metric is flat.

Of course, many different A and E fields correspond to the same metric, but they all differ by gauge transformations. So in 3 dimensions, BF

theory with gauge group $SO(2,1)$ is really just an alternate formulation of Lorentzian general relativity without matter fields at least when E is one-to-one. When E is not one-to-one, the metric g defined above will be degenerate, but the field equations of BF theory still make perfect sense. Thus 3-D BF theory with gauge group $SO(2,1)$ may be thought of as an extension of the vacuum Einstein equations to the case of degenerate metrics. If instead we take $G = SO(3)$, all these remarks still hold except that the metric g is Riemannian rather than Lorentzian when E is one-to-one. This theory is Riemannian general relativity.

We can also express general relativity in 3 dimensions as a BF theory by taking the double cover $Spin(2,1) \simeq SL(2,\mathbb{R})$ or $Spin(3) \simeq SU(2)$ as gauge group and letting P be the spin bundle. This does not affect the classical theory but it does affect the quantum theory. Nonetheless, it is very popular to take these groups as gauge groups in 3-dimensional quantum gravity.

To determine the classical phase space of BF theory we assume spacetime has the form $M = \mathbb{R} \times \Sigma$ where the real line $\mathbb{R}$ represents time and Σ is an oriented smooth $(n-1)$-dimensional manifold representing space. This is no real loss of generality, since any oriented hypersurface in any oriented n-dimensional manifold has a neighborhood of this form. We can thus use the results of canonical quantization to study the dynamics of BF theory on quite general spacetimes. If we work in temporal gauge, where the time component of the connection A vanishes, we see the momentum canonically conjugate to A is

$$E := \frac{\partial \mathcal{L}}{\partial \dot{A}}.$$

This is reminiscent of the situation in electromagnetism, where the electric field is canonically conjugate to the vector potential. This is why we use the notation 'E'. Originally people used the notation 'B' for this field[1], hence the term 'BF theory', which has subsequently become ingrained. But to understand the physical meaning of the theory, it is better to call this field E and think of it as analogous to the electric field. Of course, the analogy is best when $G = U(1)$.

Let's now see what is the relation between BF theory and gravity in 4 dimensions. The things here are quite different from the 3-dimensional case since general relativity in 4 dimensions has local degrees of freedom!

Let's remember very briefly from chapter 1 the Palatini formalism of general relativity.

Let the spacetime be given by a 4-dimensional oriented smooth manifold

[1] Just because the other field is A, so that the couple of fields of the theory were the first two letters of the alphabet!

M. We choose a bundle $\mathcal{T}$ over M that is isomorphic to the tangent bundle, but not in any canonical way.

This bundle, or any of its fibers, is called the 'internal space'. We equip it with an orientation and a metric η, either Lorentzian or Riemannian. Let P denote the oriented orthonormal frame bundle of M. This is a principal G-bundle, where G is either $SO(3,1)$ or $SO(4)$ depending on the signature of η. The basic fields in the Palatini formalism are:

- a connection A on P;
- a $\mathcal{T}$-valued 1-form e on M.

The curvature of A is an $ad(P)$-valued 2-form which, as usual, we call F. Note however that the bundle $ad(P)$ is isomorphic to the second exterior power $\bigwedge^2 \mathcal{T}$. Thus we are free to switch between thinking of F as an $ad(P)$-valued 2-form and a $\bigwedge^2 \mathcal{T}$-valued 2-form. The same is true for the field $e \wedge e$. The Lagrangian density of the theory is

$$\mathcal{L} = Tr(e \wedge e \wedge F).$$

Let's explain the meaning of the notation above: there we first take the wedge products of the differential form parts of $e \wedge e$ and F while simultaneously taking the wedge products of their 'internal' parts, obtaining the $\bigwedge^4 \mathcal{T}$-valued 4-form $e \wedge e \wedge F$. The metric and orientation on $\mathcal{T}$ give us an 'internal volume form', that is, a nowhere vanishing section of $\bigwedge^4 \mathcal{T}$. We can write $e \wedge e \wedge F$ as this volume form times an ordinary 4-form, which we call $Tr(e \wedge e \wedge F)$.

To obtain the field equations, we set the variation of the action to zero:

$$\begin{aligned} 0 &= \delta \int_M \mathcal{L} \\ &= \int_M Tr(\delta e \wedge e \wedge F + e \wedge \delta e \wedge F + e \wedge e \wedge \delta F) \\ &= \int_M Tr(2\delta e \wedge e \wedge F + e \wedge e \wedge d_A \delta A) \\ &= \int_M Tr(2\delta e \wedge e \wedge F - d_A(e \wedge e) \wedge \delta A). \end{aligned}$$

The field equations are thus

$$e \wedge F = 0, \qquad d_A(e \wedge e) = 0.$$

These equations are really just an extension of the vacuum Einstein equation to the case of degenerate metrics. To see this, first define a metric g on M by

$$g(v, w) := \eta(ev, ew).$$

When $e : TM \to \mathcal{T}$ is one-to-one, g is non-degenerate, with the same signature as η. The equation $d_A(e \wedge e) = 0$ is equivalent to $e \wedge d_A e = 0$, and when e is one-to-one this implies $d_A e = 0$. If we use e to pull back A to a metric-preserving connection Γ on the tangent bundle, the equation $d_A e = 0$ says that Γ is torsion-free, so Γ is the Levi-Civita connection of g. This lets us rewrite $e \wedge F$ in terms of the Riemann tensor. In fact, $e \wedge F$ is proportional to the Einstein tensor, so $e \wedge F = 0$ is equivalent to the vacuum Einstein equation!

There are a number of important variants of the Palatini formulation which give the same classical physics (at least for non-degenerate metrics) but suggest different approaches to quantization. Most simply, we can pick a spin structure on M and use the double cover $Spin(3,1) \simeq SL(2,\mathbb{C})$ or $Spin(4) \simeq SU(2) \times SU(2)$ as gauge group. As we have seen in chapter 1, a subtler trick is to work with the self-dual or 'lefthanded' part of the spin connection. In the Riemannian case this amounts to using only one of the $SU(2)$ factors of $Spin(4)$ as gauge group; in the Lorentzian case we need to complexify $Spin(3,1)$ first, obtaining $SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$, and then use one of these $SL(2,\mathbb{C})$ factors.

The Palatini formulation of general relativity brings out its similarity to BF theory. In fact, if we set $E := e \wedge e$, the Palatini Lagrangian density looks exactly like the BF Lagrangian density:

$$\mathcal{L}_{BF} = Tr(E \wedge F);$$

$$\mathcal{L}_{\mathrm{Pal}} = Tr(e \wedge e \wedge F).$$

The big difference, of course, is that not every $ad(P)$-valued 2-form E is of the form $e \wedge e$. This restricts the allowed variations of the E field when we compute the variation of the action in general relativity. As a result, the equations of general relativity in 4 dimensions: $e \wedge F = 0$, $d_A E = 0$ are weaker than the BF theory equations: $F = 0$, $d_A E = 0$. Another, subtler difference is that, even when E is of the form $e \wedge e$, we cannot uniquely recover e from E. In the non-degenerate case there is only a sign ambiguity: both e and $-e$ give the same E. Luckily, changing the sign of e does not affect the metric. In the degenerate case the ambiguity is greater, but we need not be unduly concerned about it, since we do not really know the correct generalization of Einsteins equation to degenerate metrics. The relation between the Palatini formalism and BF theory suggests that one develops a spin foam model of quantum gravity by taking the spin foam model for BF theory and imposing extra constraints: quantum analogues of the constraint that E be of the form $e \wedge e$!

Let's analyze how to impose this constraint in the Riemannian case, i.e. when we pick a spin structure for spacetime and take the double cover $Spin(4)$ as our gauge group. Locally we may think of the E field as taking values in the Lie algebra $\mathfrak{so}(4)$, but the splitting $\mathfrak{so}(4) \simeq \mathfrak{so}(3) \oplus \mathfrak{so}(3)$ lets us write E as the sum of left-handed and right-handed parts E taking values in $\mathfrak{so}(3)$. If $E = e \wedge e$, the following constraint holds for all vector fields v, w on M: $\|E^{+}(v,w)\| = \|E^{-}(v,w)\|$ where $\| \ \|$ is the norm on $\mathfrak{so}(3)$ coming from the Killing form. This constraint is sufficient to guarantee that E is of the form $e \wedge e$.

Now that we have finally written this constraint at the classical level, we must describe the spin foam models of gravity and then show how to impose this constraint in the quantum theory. We do this in the next sections.

6.3 Dynamical triangulations and spin foam models for quantum gravity

In the previous chapters we have developed canonical quantization of gravity by using the concept of *graph*. However there is another way to perform a canonical quantization that can be thought as a 'dual' version of the previous one. This procedure is intimately connected to the triangulations of manifolds and now the role played by the graph is played by an object called *piecewise linear cell complex*.

We want to give a very brief presentation of the quantization via dynamical triangulations, this will serve as a motivation to construct a spin foam model starting from triangulations.

The reference is [3]. Let M be an n-dimensional, ($n \geq 2$), manifold of given topology. Very naively, to triangulate M corresponds to subdivide it in a finite number of submanifolds M_n with $(n-1)$-dimensional boundaries Σ_k, $k = 1, 2, \ldots$ Let $\mathrm{Riem}(M)$, $\mathrm{Lor}(M)$ and $\mathrm{Diff}(M)$ respectively denote the space of Riemannian (Lorentzian) metrics g on M, and the group of diffeomorphisms on M.

To fix the ideas let's stuck on the Riemannian case. In the continuum formulation of quantum gravity the task is to perform a path integral over equivalence classes of metrics to get a partition function[2]:

$$Z(g, \Sigma_k) = \sum_{Top(M)} \int_{\mathrm{Riem}(M)/\mathrm{Diff}(M)} \mathcal{D}[g] e^{-S(g,\Sigma_k)}$$

where $S(g, \Sigma_k)$ is the Einstein-Hilbert action associated with the Riemannian manifold (M, g) in which there appear boundary terms depending on the extrinsic curvature induced on the boundaries Σ_k. These terms are such that, if we glue together any two manifolds (M_1, g_1) and (M_2, g_2) along a common boundary Σ, we get $S(g_1, \Sigma) + S(g_2, \Sigma) = S(g_1 + g_2, \Sigma)$. In this way, the partition function satisfies the basic composition law which describes how M can interpolate between two fixed boundaries Σ_1 and Σ_2 by summing over all possible intermediate states.

The formal measure $\mathcal{D}[g]$ that characterizes such sort of path integration should satisfy some basic properties, in particular it should be defined on the space $\mathrm{Riem}(M)/\mathrm{Diff}(M)$ to avoid counting as distinct any two Riemannian metrics g_1 and g_2 which differ one from the other simply by the action of a diffeomorphism of $\phi : M \to M$ such that $g_2 = \phi^* g_1$.

The hope behind the dynamical triangulation algorithm of quantum gravity (or *simplicial quantum gravity* for reasons that will be cleared later) is that some of the well known problems concerning the characterization of the path-integral measure can be properly addressed, in a non-perturbative setting, by approximating the path integration over inequivalent Riemannian structures with a summation over combinatorially equivalent objects called piecewise linear (PL in the following) manifolds.

The first attempt of using PL geometry in relativity dates back to the pioneering work of Regge [80]. His proposal was to approximate Riemannian structures by PL-manifolds in such a way as to obtain a coordinate-free formulation of general relativity. The basic observation in this approach is that parallel transport and the (integrated) scalar curvature have natural counterparts on PL manifolds once one gives consistently the lengths of the links of the triangulation defining the PL structure. The link length is the dynamical variable in Regge calculus, and classically the PL version of the Einstein field equations is obtained by fixing a suitable triangulation and by varying the length of the links so as to find the extremum of the Regge action. If the original triangulation is sufficiently fine, this procedure consistently provides a good approximation to the smooth spacetime manifold which is the corresponding smooth solution of the Einstein equations.

[2]The partition function in quantum field theory is a very important object, in fact it enables to compute the transition amplitude between two quantum states, i.e. the probability that a physical system transforms from a given state to another, see [31] for a wider discussion, or [52] for the genesis of the partition function in statistical field theory.

Now we list the mathematically rigorous definitions of the objects appearing in the theory of triangulations (the interested reader is referred to [85] for a wider discussion).

Def. 6.3.1 *A subset $X \subseteq \mathbb{R}^n$ is said to be a* **polyhedron** *if every point $x \in X$ has a neighborhood in X of the form*

$$\{\alpha x + \beta y \mid \alpha, \beta \geq 0,\ \alpha + \beta = 1,\ y \in Y,\ Y \subseteq X,\ Y \text{compact}\}.$$

A compact convex polyhedron X for which the smallest affine space containing X is of dimension k is called a **k-cell***.*

A k-cell X is **oriented** *if X minus the union of its proper faces, thought of as a k-dimensional manifold, is equipped with an orientation.*

For example, $\mathbb{R}^n$ is a polyhedron and any open subset of a polyhedron is a polyhedron. Cells, on the other hand, are more special. For example

- every 0-cell is a *point*;
- every 1-cell is a compact *interval* affinely embedded in $\mathbb{R}^n$;
- every 2-cell is a convex compact *polygon* affinely embedded in $\mathbb{R}^n$.

The 'vertices' and 'faces' of a cell X are defined as follows.

Def. 6.3.2 *Given a point $x \in X$, let $< x, X >$ be the union of lines L through x such that $L \cap X$ is an interval with x in its interior. If there are no such lines, we define $< x, X >$ to be $\{x\}$ and call x a* **vertex of** *X. One can show that $< x, X > \cap X$ is a cell, and such a cell is called a* **face of** *X.*

One can show that any cell X has finitely many vertices v_i and that X is the convex hull of these vertices, meaning that:

$$X = \left\{ \sum \alpha_i v_i \mid \alpha_i \geq 0,\ \sum \alpha_i = 1 \right\}.$$

Similarly, any face of X is the convex hull of some subset of the vertices of X. However, not every subset of the vertices of X has a face of X as its convex hull. If the cell Y is a face of X we write $Y \leq X$. This relation is transitive, and if $Y, Y' \leq X$ we have $Y \cap Y' \leq X$.

Def. 6.3.3 *A* **piecewise linear cell complex***, or* **complex** *for short, is a collection κ of cells in some $\mathbb{R}^n$ such that:*

1. *if $X \in \kappa$ and $Y \leq X$ then $Y \in \kappa$;*

2. if $X, Y \in \kappa$, then $X \cap Y \leq X, Y$.

So, in particular, 2-dimensional complexes can be intuitively thought as a finite collection of polygons attached to the other along their edges. A complex is **k-dimensional** if it has cells of dimension k but no higher. A complex is **oriented** if every cell is equipped with an orientation, with all 0-cells being equipped with the positive orientation. The union of the cells of a complex κ is a polyhedron which we denote by $|\kappa|$.

Def. 6.3.4 *An n-**simplex** $\sigma^n \equiv (x_0, \dots, x_n)$ **with vertices** $x_0, \dots, x_n$ is the following subspace of $\mathbb{R}^d$, (with $d > n$):*

$$\left\{ \sum_{i=0}^{n} \lambda_i x_i,\ x_0, \dots, x_n \in \mathbb{R}^d,\ \lambda_i \geq 0,\ \sum_{i=0}^{n} \lambda_i = 1 \right\}.$$

*A **face** of a n-simplex σ^n is any simplex whose vertices are a subset of those of σ^n and a **simplicial complex** K is a finite collection of simplices in $\mathbb{R}^d$ such that if $\sigma_1^n, \sigma_2^m \in K$ then so are all of its faces, and if $\sigma_1^n, \sigma_2^m \in K$ then $\sigma_1^n \cap \sigma_2^m$ is either a face of σ_1^n or is empty. The h-**skeleton** of K is the subcomplex $K_h \subset K$ consisting of all simplices of K of dimension $\leq h$.*

*A **piecewise linear manifold** of dimension n is a polyhedron $M = |\kappa|$ each point of which has a neighborhood in M piecewise linear homeomorphic to an open subset of $\mathbb{R}^n$.*

*A **triangulated manifold** can be characterized as its underlying piecewise linear manifold.*

The link between triangulation and canonical loop quantum gravity is that **every 1-dimensional oriented complex has an underlying graph, that uniquely determines the 1-dimensional complex up to piecewise linear homeomorphisms**.

Since spin networks are based on graphs, and since they describe 3-quantum geometry, this result implies that 3-geometry can be also described in terms of spin networks based on a 1-dimensional complex. But then one can 'grow up' to dimension 4 to describe 4-quantum geometry by considering a new object of the theory based on 2-dimensional complexes. This object is precisely what we call a spin foam!

We now give the precise definitions of spin networks and spin foams in terms of complexes.

First of all let's fix the notations concerning 1-dimensional oriented complexes. Such a complex has a set V of 0-cells or **vertices**, and also a set E of oriented 1-cells or **edges**. The orientation on each edge $e \in E$ picks out one of its endpoints as its source $s(e) \in V$ and the other as its target $t(e) \in V$.

If v is the source of e we say e **is outgoing from** v, while if v is the target of e we say that e **is incoming to** v.

Def. 6.3.5 (Spin network in terms of 1-dim. complexes) *A spin network Ψ is a triple $(\gamma, \vec{\rho}, \vec{I})$ consisting of:*

1. *a 1-dimensional oriented complex γ;*
2. *a labelling $\vec{\rho}$ of each edge e of γ by a unitary irreducible representation ρ_e of G;*
3. *a labelling $\vec{\jmath}$ of each vertex v of γ by an intertwiner:*

$$I_v : \rho_{e_1} \otimes \cdots \otimes \rho_{e_n} \to \rho_{e'_1} \otimes \cdots \otimes \rho_{e'_m}$$

   where $e_1, \ldots, e_n$ are the edges incoming to v and $e'_1, \ldots, e'_m$ are the edges outgoing from v.

Now we grow up in dimension and define the spin foams. A 2-dimensional oriented complex has a finite set of vertices V, a finite set of edges E, and finite sets of n-sided 2-cells or **faces** F_n for each $n \geq 3$, with only finitely many F_n being nonempty. As in a 1-dimensional oriented complex, the orientations of the edges give maps

$$s, t : E \to V$$

assigning to each edge its source and target. In addition, the orientation on any 2-cell $f \in F_n$ puts a cyclic ordering on its faces and vertices. Suppose we arbitrarily choose a 'first' vertex for each 2-cell f of our complex. Then we may number all its vertices and edges from 1 to n. It is convenient to think of these numbers as lying in $\mathbb{Z}_n$. We thus obtain maps

$$e_i : F_n \to E, \qquad v_i : F_n \to V \qquad\qquad i \in \mathbb{Z}_n.$$

Note that for each $f \in F_n$ either

$$s(e_i(f)) = v_i(f) \ \textit{and} \ t(e_i(f)) = v_{i+1}(f) \tag{6.1}$$

or

$$t(e_i(f)) = v_i(f) \ \textit{and} \ s(e_i(f)) = v_{i+1}(f). \tag{6.2}$$

If (6.1) holds, we say f is incoming to e, while if (6.2) holds, we say f is outgoing from e. In other words, f is incoming to e if the orientation of e agrees with the orientation it inherits from f, while it is outgoing if these orientations do not agree.

First we define a special class of spin foams.

Def. 6.3.6 *A closed spin foam F is a triple $(\kappa, \vec{\rho}, \vec{I})$ consisting of:*

1. *a 2-dimensional oriented complex κ;*
2. *a labelling $\vec{\rho}$ of each face f of κ by an irreducible representation ρ_f of G;*
3. *a labelling $\vec{I}$ of each edge e of κ by an intertwiner*

$$I_e : \rho_{f_1} \otimes \cdots \otimes \rho_{f_n} \to \rho_{f'_1} \otimes \cdots \otimes \rho_{f'_m}$$

where $f_1, \ldots, f_n$ are the faces incoming to e and $f'_1, \ldots, f'_m$ are the faces outgoing from e.

Next we turn to general spin foams. In general, a spin foam $F : \Psi \to \Psi'$ will go from a spin network Ψ to a spin network Ψ'. It has 'free edges', the edges of the spin networks Ψ and Ψ', which are not labelled by intertwiners. It also has edges ending at the spin network vertices, and the intertwiners labelling these edges must match those labelling the spin network vertices. A closed spin foam is just a spin foam of the form $F : \emptyset \to \emptyset$, where $\emptyset$ is the *empty spin network*: the spin network with no vertices and no edges.

To make this more precise, suppose γ is a 1-dimensional oriented complex and κ is a 2-dimensional oriented complex. Note that the product $\gamma \times [0, 1]$ becomes a 2-dimensional oriented complex in a natural way. We say γ borders κ if there is a one-to-one affine map $c : |\gamma| \times [0, 1] \to |\kappa|$ mapping each cell of $\gamma \times [0, 1]$ onto a unique cell of κ in an orientation-preserving way, such that c maps $\gamma \times [0, 1)$ onto an open subset of $|\kappa|$. Note that in this case, c lets us regard each j-cell of γ as the face of a unique $(j + 1)$-cell of κ.

Each vertex v of γ is the source or target of a unique edge of κ, which we denote by $\tilde{v}$, and each edge e of γ is the edge of a unique face of κ, which we denote by $\tilde{e}$. It is easier to first define spin foams $F : \emptyset \to \Psi$ and then deal with the general case.

Def. 6.3.7 *Suppose that $\Psi = (\gamma, \vec{\rho}, \vec{I})$ is a spin network. A spin foam $F : \emptyset \to \Psi$ is a triple $(\kappa, \tilde{\rho}, \tilde{I})$ consisting of:*

1. *a 2-dimensional oriented complex κ such that borders κ;*
2. *a labelling $\tilde{\rho}$ of each face f of κ by an irreducible representation $\tilde{\rho}_f$ of G;*
3. *a labelling $\tilde{I}$ of each edge e of κ not lying in γ by an intertwiner*

$$\tilde{I}_e : \rho_{f_1} \otimes \cdots \otimes \rho_{f_n} \to \rho_{f'_1} \otimes \cdots \otimes \rho_{f'_m}$$

where $f_1, \ldots, f_n$ are the faces incoming to e and $f'_1, \ldots, f'_m$ are the faces outgoing from e;

such that the following hold:

$(\cdot)$ *for any edge* e *of* γ, $\tilde{\gamma}_{\tilde{e}} = \rho_e$ *if* $\tilde{e}$ *is incoming to* e, *while* $\tilde{\gamma}_{\tilde{e}} = (\rho_e)^*$ *if* $\tilde{e}$ *is outgoing to* e*;*

$(\cdot\ \cdot)$ *for any vertex* v *of* γ, $\tilde{I}_{\tilde{e}}$ *equals* I_e *after appropriate dualizations.*

To define general spin foams, we need the notions of 'dual' and 'tensor product' for spin networks. Suppose $\Psi = (\gamma, \vec{\rho}, \vec{I})$ is a spin network, then the dual of is the spin network Ψ^* with the same underlying 1-dimensional oriented complex, but with each edge e labelled by the representation ρ_e^* and with each vertex v labelled by the appropriately dualized form of the the intertwining operator I_v.

Suppose that $\Psi = (\gamma, \vec{\rho}, \vec{I})$ and $\Psi' = (\gamma, \vec{\rho'}, \vec{I'})$ are disjoint spin networks in the same space $\mathbb{R}^n$. Then the *tensor product* $\Psi \otimes \Psi'$ is defined to be the spin network whose underlying 1-dimensional oriented complex is the disjoint union of γ and γ', with edges and vertices labelled by representations and intertwiners using ρ, ρ' and I, I'.

Def. 6.3.8 *Given disjoint spin networks* Ψ *and* Ψ' *in* $\mathbb{R}^n$ *a spin foam* $F : \Psi \to \Psi'$ *is defined to be a spin foam* $F : \emptyset \to \Psi^* \otimes \Psi'$.

Now we would obviously like to implement the abstract spin foam model presented above in an explicit way in order to reach a spin foam model of quantum geometry. In the past years many of such implementations has been proposed, but the most promising one seems to be the so-called Barrett-Crane model.

We describe the Barrett-Crane model for Riemannian quantum gravity in the next section and we reserve the discussion of the model for Lorentzian quantum gravity in the next chapter, since it requires a more sophisticated mathematical analysis.

6.4 Spin foams in lattice gauge theory

As we said many times, while spin networks describe states, spin foams describe 'histories': the path integral can be computed as a sum over spin foams. In this context we work, not with the abstract spin networks of the previous section, but with spin networks embedded in a manifold representing space. Similarly, we work with spin foams embedded in a manifold representing spacetime. Throughout the rest of the chapter we assume these manifolds are oriented and equipped with a fixed triangulation.

The triangulation specifies another decomposition of the manifold into cells called the 'dual complex'. There is a one-to-one correspondence between k-simplices in the triangulation of an n-manifold and $(n-k)$-cells in the dual complex, each k-simplex intersecting its corresponding dual $(n-k)$-cell in a single point. Our spin networks and spin foams live in the appropriate dual complexes. We need to work with oriented complexes, so we orient each cell of the dual complex in an arbitrary fashion.

We begin by recalling how spin networks describe states in lattice gauge theory. We fix a compact connected Lie group G as our gauge group, and suppose S is an $(n-1)$-manifold representing *space*. We assume S is equipped with a triangulation Δ and a principal G-bundle $P \to S$. Also, we choose a trivialization of P over every 0-cell of the dual complex Δ^*. Remember that the **k-skeleton** of a complex is the subcomplex formed by all cells of dimension less than or equal to k.

In particular, **the 1-skeleton of Δ^* is a graph, and we can set up gauge theory on this graph in the usual way**: we represent parallel transport along each edge of Δ^* as an element of G, so we define the space of connections[3] $\mathcal{A}_S$ by

$$\mathcal{A}_S := \prod_{e \in \Delta_1^*} G$$

where Δ_k^* denotes the set of k-cells in Δ^*.

Similarly, we represent a gauge transformation at each vertex of Δ^* as an element of G, so we define the group of gauge transformations

$$\mathcal{G}_S := \prod_{v \in \Delta_0^*} G.$$

The group $\mathcal{G}_S$ acts on $\mathcal{A}_S$, and the quotient space $\mathcal{A}_S/\mathcal{G}_S$ is the space of connections modulo gauge transformations in this setting. The space $\mathcal{A}_S$ has a measure on it given by a product of copies of Haar measures, and this measure pushes forward to a measure on $\mathcal{A}_S/\mathcal{G}_S$.

Using this measure we are able to define the Hilbert space $L^2(\mathcal{A}_S/\mathcal{G}_S)$.

Suppose that $\Psi = (\gamma, \rho, I)$ is a spin network[4] in S, where γ is the 1-skeleton of Δ^*. Then we know that it defines a state in $L^2(\mathcal{A}_S/\mathcal{G}_S)$, which we also call Ψ. Moreover, such spin network states span $L^2(\mathcal{A}_S/\mathcal{G}_S)$. In fact, we obtain an orthonormal basis of states as ρ ranges over all labellings of the edges of Δ^* by irreducible representations of G and I ranges over all labellings of the vertices by intertwiners chosen from some orthonormal basis.

[3] In lattice gauge theory generalized connections and ordinary connections coincide. The same is true for gauge transformations.

[4] For simplicity in this section we suppress the vector symbol over ρ and I.

Next we turn to spin foams. For this, suppose S is a submanifold of some n-manifold M representing spacetime. We assume that $P \to M$ is a principal G-bundle over M that restricts to the already given bundle over S, and that Θ is a triangulation of M that restricts to the already given triangulation of S.

We wish to see how a closed spin foam in M determines a spin network in S. Suppose that $F = (\kappa, \tilde{\rho}, \tilde{I})$ is a closed spin foam in M, that is, one for which κ is the 2-skeleton of Θ^*.

Note that every k-cell X of Θ^* is contained in a unique $(k+1)$-cell $\tilde{X}$ of Θ^*. Since S and M are oriented, the normal bundle of S acquires an orientation. Similarly, since X and $\tilde{X}$ are oriented, the normal bundle of the interior of X in $\tilde{X}$ acquires an orientation. But this latter bundle can be identified with the restriction of the normal bundle of S to the interior of X, so we have two different orientations on the same bundle. We say that $\tilde{X}$ *is incoming to* X if these orientations agree, and *outgoing to* X if they do not.

We thus obtain a spin network in S as follows.

Theorem 6.4.1 *If $F = (\kappa, \tilde{\rho}, \tilde{I})$ is a closed spin foam in M, then there exists a unique spin network $F|_S = (\gamma, \rho, I)$ in S such that:*

1. *γ is the 1-skeleton of Δ^*;*
2. *for any edge e of γ, $\tilde{\rho}_{\tilde{e}} = \rho_e$ if $\tilde{e}$ is incoming to e, while $\tilde{\rho}_{\tilde{e}} = (\rho_e)^*$ if $\tilde{e}$ is outgoing to e;*
3. *if v is a vertex of γ, then $\tilde{I}_{\tilde{e}}$ equals I_e after appropriate dualizations.*

The proof is trivial but the physical idea is important: **a history on spacetime determines a state on any submanifold corresponding to space at a given time**.

Any spin foam $F : \Psi \to \Psi'$ in M determines an operator from $L^2(\mathcal{A}_S/\mathcal{G}_S)$ to $L^2(\mathcal{A}_{S'}/\mathcal{G}_{S'})$, which we also denote by F, such that

$$< \Phi'|F\Phi >=< \Phi'|\Psi' >< \Psi|\Phi >$$

for any states Φ, Φ'. The point is that the spin foam F represents a history going from the initial state Ψ to the final state Ψ', and the corresponding operator does not depend on the behavior of this history at 'intermediate times', that is, in the interior of M. We call the operator F a **spin foam operator**.

Just as the space of states $L^2(\mathcal{A}_S/\mathcal{G}_S)$ is spanned by spin network states, every operator from $L^2(\mathcal{A}_S/\mathcal{G}_S)$ to $L^2(\mathcal{A}_{S'}/\mathcal{G}_{S'})$ will be a linear combination of spin foam operators if there is a spin foam $F : \Psi \to \Psi'$ for every pair of spin networks Ψ and Ψ'.

BF theory with gauge group $SU(2)$ is 3-dimensional Euclidean quantum gravity, and the lattice formulation of this theory is just the Ponzano-Regge model [75]. When the cosmological constant Λ is nonzero, one obtains instead the 'regularized' model of Turaev-Viro [91], where the quantum group $SU_q(2)$ takes the place of the gauge group $SU(2)$, with the deformation parameter q being a function of Λ. Even if the use of quantum groups provides a regularization of many mathematical problems, the physical significance of a theory formulated in terms of quantum groups is still obscure.

In the next section we concentrate on a more physical, but also more complicated example, namely 4-dimensional Euclidean quantum gravity.

6.5 The Barret-Crane model for Riemannian quantum gravity

We now turn to the question of how spin foams describe quantum 4-geometries. Let M be a piecewise-linear 4-manifold equipped with a triangulation Θ. We also assume there is a principal bundle $\tilde{P} \to M$ with structure group $Spin(4)$. Using this bundle we can do lattice gauge theory over M as explained in the previous section. In particular, we can consider a $Spin(4)$ spin foam in M and try to understand it as equipping M with a 'quantum 4-geometry'.

We begin by fixing a particular spin foam in M and seeing how far we can get. One can understand quite a bit about the geometry of a 4-manifold by studying the geometry of 3-manifolds embedded in it. Thus we begin by considering 3-dimensional submanifolds of M. Suppose we have an oriented 3-dimensional piecewise-linear submanifold $S \subset M$ with a triangulation $\Delta \subset \Theta$.

Since the group $Spin(4)$ is a product of two copies of $SU(2)$, the bundle $\tilde{P}$ is a product of 'left-handed' and 'right-handed' $SU(2)$ bundles P^+ and P^-. This implies that the space $\mathcal{A}_S$ of connections on $\tilde{P}\Big|_S$ can be written as a product $\mathcal{A}_S = \mathcal{A}_S^+ \times \mathcal{A}_S^-$ where $\mathcal{A}_S^\pm$ is the space of connections on $P^\pm|_S$, and similarly

$$\mathcal{A}_S/\mathcal{G}_S = \mathcal{A}_S^+/\mathcal{G}_S^+ \times \mathcal{A}_S^-/\mathcal{G}_S^-$$

where $G_S^\pm$ is the space of gauge transformations on $P^\pm|_S$. Thus we have

$$L^2(\mathcal{A}_S/\mathcal{G}_S) \simeq L^2(\mathcal{A}_S^+/\mathcal{G}_S^+) \otimes L^2(\mathcal{A}_S^-/\mathcal{G}_S^-)$$

where both the 'left-handed' and 'right-handed' factors in this tensor product are spanned by $SU(2)$ spin networks in the dual 1-skeleton of S. In particular, both factors are isomorphic.

In what follows we will arbitrarily choose to favor $L^2(\mathcal{A}_S^+/\mathcal{G}_S^+)$ in certain constructions, influenced by the Ashtekar's strategy of quantizing gravity using the left-handed (or self-dual) spin connection. However, the physics is independent of this choice.

We can define various geometrically interesting operators on the left-handed Hilbert space $L^2(\mathcal{A}_S^+/\mathcal{G}_S^+)$. We can tensor these with the identity on the right-handed Hilbert space to obtain corresponding operators on $L^2(\mathcal{A}_S/\mathcal{G}_S)$. This allows us to interpret states in the latter space as quantum 3-geometries. Moreover, as described previously, any spin foam F in M determines a state $F|_S \in L^2(\mathcal{A}_S/\mathcal{G}_S)$. Thus spin foams in M determine quantum 3-geometries for S.

More concretely, note that a spin foam in M is a labelling of each 2-cell in the dual complex Θ^* by an irreducible representation of $Spin(4)$ together with a labelling of each 1-cell by an intertwining operator. The 1-cells and 2-cells in the dual complex correspond to tetrahedra and triangles in the triangulation Θ, respectively. Moreover, any irreducible representation of $Spin(4)$ is of the form $j^+ \otimes j^-$ where j^+ and j^- are representations of the left-handed and right-handed copies of $SU(2)$. Thus if we pick a splitting of each tetrahedron in Θ, a spin foam in M amounts to a labelling of each triangle and each tetrahedron in Θ by a pair of spins.

As explained above, given any triangle f in the triangulation of S there is an area operator $\hat{A}_f^+$ on $L^2(\mathcal{A}_S/\mathcal{G}_S)$ coming from the area operator on the left-handed Hilbert space $L^2(\mathcal{A}_S^+/\mathcal{G}_S^+)$. We define the expectation value of the area of f in the spin foam F to be

$$< F, \hat{A}_f^+ >=< F|_S \, , \hat{A}_f^+ \, F|_S > .$$

Note that any triangle f in Θ lies in some submanifold $S \subseteq M$ of the form we are considering, and the above quantity is independent of the choice of S. Thus the expectation value of the area of any triangle in M is well-defined in the quantum 4-geometry described by any spin foam. The space S serves only as a disposable tool for studying the geometry of the spacetime M. Note that we can easily extend the above formula to formal linear combinations of spin foams. This allows us to think of A_f^+ as an operator on the space of formal linear combinations of spin foams. More generally, we can define an area operator $\hat{A}_\Sigma^+$ for any 2-dimensional submanifold $\Sigma \subseteq M$ built from triangles in Θ by adding up the area operators for the triangles it contains. Similarly, for every tetrahedron T in the triangulation of S there is a 3-volume operator $\hat{V}_T^+$ on $L^2(\mathcal{A}_S/\mathcal{G}_S)$, and we can define the expectation value of the 3-volume of T in the spin foam F to be

$$< F|\hat{V}_T^+ F >=< F|_S\, |\hat{V}_T^+\, F|_S > .$$

Again, the right-hand side is actually independent of S, since it depends only on the data by which F labels the tetrahedron T and its faces. As with the area of a triangle, we can also extend V_T to an operator on the space of formal linear combinations of spin foams. More generally, we can define the volume operator $\hat{V}_S^+$ for any 3-dimensional submanifold $S \subseteq M$ built from tetrahedra in Θ to be the sum of volume operators for the tetrahedra it contains.

So, where do we stand? We have seen that the spin foams endow each such submanifold with a quantum 3-geometry. However, for these quantum 3-geometries to fit together to form a sensible quantum 4-geometry, it appears that certain constraints must hold. Following the ideas of Barrett and Crane [24], we arrive at these constraints through a study of the 'quantum 4-simplex'. Here we take S to be the boundary of a single 4-simplex in M. A spin foam in M gives each of the ten triangles in S an area and each of the five tetrahedra in S a volume. However, the geometry of a 4-simplex affinely embedded in Euclidean $\mathbb{R}^4$ is determined by only ten numbers, e.g., the lengths of its edges. This suggests that some constraints must hold for S to be the boundary of a 'flat' 4-simplex. We wish the 4-simplices in M to be flat because we want a picture similar to that of the Regge calculus, where spacetime is pieced together from flat 4-simplices, and curvature is concentrated along their boundaries [80].

Interestingly, these constraints are the same that arise naturally from the relationship between general relativity and BF theory as we have described in section 6.2. In fact doing general relativity with the cotetrad field e (as in Palatini formulation) is very much like describing 4-simplices using vectors for edges, while doing general relativity with the E field constrained to be of the form $E = e \wedge e$ is very much like describing 4-simplices using bivectors[5] for faces. Suppose we have a 4-simplex affinely embedded in $\mathbb{R}^4$. We can number its vertices $0, 1, 2, 3, 4$ and translate it so that the vertex 0 is located at the origin. Then one way to describe its geometry is by the positions e_1, e_2, e_3, e_4 of the other four vertices. Another way is to use the bivectors

$$E_{ab} = e_a \wedge e_b \in \Lambda^2\mathbb{R}^4$$

corresponding to the six triangular faces with 0 as one of their vertices. However, not every collection of bivectors E_{ab} comes from a 4-simplex this

[5] A **bivector** in n dimensions is simply an element of $\Lambda^2\mathbb{R}^n$.

way. In addition to the obvious skew-symmetry $E_{ab} = -E_{ba}$, some extra constraints must hold. The remarkable fact is that **these constraints have exactly the same form as the extra constraints that we need to obtain general relativity from BF theory!**

Barrett and Crane showed that these constraints take a particularly nice form if we describe them in terms of bivectors corresponding to all ten triangular faces of the 4-simplex. Quantizing these constraints, one obtains conditions that a $Spin(4)$ spin foam must satisfy to describe a quantum 4-geometry built from flat quantum 4-simplices. Interestingly, these conditions also guarantee that the quantum geometries for 3-dimensional submanifolds obtained using the right-handed copy of $SU(2)$ agree with those coming from the left-handed copy!

To see how this works we begin by studying the geometry of an ordinary classical 4-simplex in Euclidean $\mathbb{R}^4$. Let S be a 4-simplex with vertices $0, 1, 2, 3, 4$. As said above, we can think of a 4-simplex in $\mathbb{R}^4$ with one vertex at the origin as an affine map from S to $\mathbb{R}^4$ sending the vertex 0 to the origin. Such a map is given by fixing vectors $e_1, e_2, e_3, e_4 \in \mathbb{R}^4$ corresponding to the edges $\overline{01}, \overline{02}, \overline{03}$ and $\overline{04}$. These data amounts to a cotetrad, a linear map from the tangent space of the vertex 0 of S to $\mathbb{R}^4$. Alternatively, we can describe the 4-simplex by associating bivectors to its triangular faces. For any pair $a, b = 1, 2, 3, 4$ we can define a bivector

$$E_{ab} = e_a \wedge e_b$$

corresponding to the face $\widehat{0ab}$. However, not every collection of bivectors E_{ab} is of the form $e_a \wedge e_b$ for some vectors $e_a \in \mathbb{R}^4$. In addition to the obvious antisymmetry in the indices a and b, the following constraints are also necessary:

$$E_{ab} \wedge E_{cd} = 0 \ \text{ if } \ \{a, b\} \cap \{c, d\} \neq 0 \tag{6.3}$$

and

$$E_{12} \wedge E_{34} = E_{13} \wedge E_{42} = E_{14} \wedge E_{23}. \tag{6.4}$$

The conditions (6.3) and (6.4) are not enough. In fact, in the generic case, when the E_{ab} $(a < b)$ are linearly independent, it can be shown [29] that the conditions above admit exactly three sorts of solutions in addition to those of the form $E_{ab} = e_a \wedge e_b$, for some basis e_a of $\mathbb{R}^4$.

The first is to take

$$E_{ab} = -e_a \wedge e_b,$$

for some basis e_a, the second is to take

$$E_{ab} = \sum_{a,b=1}^{4} \epsilon_{abcd}\, e_c \wedge e_d,$$

and the third is to take

$$E_{ab} = -\sum_{a,b=1}^{4} \epsilon_{abcd}\, e_c \wedge e_d.$$

If E_{ab} can be written in any of these four ways, it can be written so uniquely up to a parity transformation $e_a \mapsto -e_a$. In what follows, we simply ignore these subtleties and **take any collection of bivectors E_{ab} antisymmetric in a, b and satisfying (6.3) and (6.4) as an adequate substitute for a cotetrad**.

Barrett and Crane make the all-important observation that conditions (6.3) and (6.4) can be rewritten in a simpler way if we use bivectors for all triangular faces of the 4-simplex, not just those having 0 as a vertex. For this, it is convenient to set $e_0 = 0$ and define

$$E_{abc} = (e_c - e_b) \wedge (e_b - e_a).$$

The bivector E_{abc} corresponds to the triangular face $\widehat{abc}$; in particular, we have $E_{\widehat{0bc}} = E_{bc}$.

The bivectors E_{abc} satisfy three sorts of constraints:

1. first, E_{abc} is *totally antisymmetric* in the indices a, b, c;

2. second, the bivectors E_{abc} $(0 \leq a < b < c \leq 4)$ satisfy five *closure constraints* of the form:

$$E_{abc} - E_{abd} + E_{acd} - E_{bcd} = 0 \qquad a < b < c < d \tag{6.5}$$

   one for each tetrahedral face of S;

3. third, using these linear constraints, the conditions (6.3) and (6.4) can be rewritten as the *quadratic constraints*

$$E_{abc} \wedge E_{a'b'c'} = 0,$$

   which hold whenever the triangles abc and $a'b'c'$ share at least one edge, that is, when they either share one edge or are the same.

The quadratic constraints $E_{abc} \wedge E_{abc} = 0$ have a particularly nice geometric interpretation: they say the bivectors E_{abc} can be written as wedge products of vectors in $\mathbb{R}^4$. Before attempting to quantize these constraints, we begin by quantizing a single bivector in 4 dimensions.

Using a metric on $\mathbb{R}^4$ we identify $\Lambda^2\mathbb{R}^4$ with $\mathfrak{so}(4)^*$, which as the dual of a Lie algebra can be thought of as a classical phase space equipped with the

Kirillov-Kostant Poisson structure in which the Poisson brackets of any to such linear functionals $f, g \in \mathfrak{so}(4)^*$ is given by $\{f, g\} := [f, g]$ (see [47] for more details).

$\mathfrak{so}(4)$ is isomorphic to $\mathfrak{so}(3) \oplus \mathfrak{so}(3)$ and $\mathfrak{so}(3)$ can be quantized following the prescriptions of *geometric quantization* of Kirillov and Kostant; it turns out that only the 'integral' coadjoint orbits contribute, that is, those for which the symplectic 2-form divided by 2π defines an integral cohomology class. These are spheres of radius $j = 0, \frac{1}{2}, 1, \ldots$, so we when we quantize the phase space $\mathfrak{so}(4)^*$ we obtain the Hilbert space of a quantum bivector in four dimensions as:

$$\mathcal{H}^+ \otimes \mathcal{H}^- \simeq \bigoplus_{j^+, j^-} j^+ \otimes j^-.$$

Note that this space is not a representation of $SO(4)$, but only of its universal cover, $Spin(4)$. The Hilbert spaces $\mathcal{H}^+$ and $\mathcal{H}^-$ correspond to the self-dual and anti-self-dual parts of the quantum bivector, either of which can be identified with a quantum bivector in three dimensions. To see this, recall that given a metric and orientation on $\mathbb{R}^4$, we may define the Hodge star operator $* : \Lambda^2\mathbb{R}^4 \to \Lambda^2\mathbb{R}^4$. The $+1$ and -1 eigenspaces of this operator are called the spaces of self-dual and anti-self-dual bivectors, respectively:

$$\Lambda^2\mathbb{R}^4 = \Lambda^2_+\mathbb{R}^4 \oplus \Lambda^2_-\mathbb{R}^4.$$

This allows us to decompose a bivector E in 4 dimensions into a self-dual part E^+ and an anti-self-dual part E^-. Under the isomorphism between $\Lambda^2\mathbb{R}^4$ and $\mathfrak{so}(4)^*$, this splitting corresponds to

$$\mathfrak{so}(4)^* \simeq \mathfrak{so}(3)^* \oplus \mathfrak{so}(3)^*.$$

Thus when we quantize $\mathfrak{so}(4)^*$ with its Kirillov-Kostant Poisson structure, we get a tensor product of Hilbert spaces corresponding to the self-dual and anti-self-dual parts of the space of bivectors.

Now we turn to quantizing the constraints that say when ten bivectors correspond to the faces of a flat 4-simplex. We would like to interpret the closure constraints as imposing gauge-invariance. The closure constraints in equation (6.5) have minus signs, the reason for these signs is obvious: in our definition of the E_{abc}, we implicitly choose orientations so that two triangular faces of each tetrahedron are oriented clockwise and two are oriented counterclockwise.

To get all plus signs in the constraint we explode Δ into a disjoint union of five tetrahedra, thus doubling the number of triangles. We can then define a bivector for each face, orienting all the faces counterclockwise. This amounts to working with 20 bivectors $E_i(T)$, where T ranges over the five tetrahedra in Δ and $i = 0, 1, 2, 3$ labels the faces of each tetrahedron.

In these terms the closure constraints become

$$E_0(T) + E_1(T) + E_2(T) + E_3(T) = 0.$$

To quantize the closure constraints this way, we start with a Hilbert space

$$\bigotimes_{T \in \Delta_3} \bigotimes^4 (\mathcal{H}^+ \otimes \mathcal{H}^-)$$

describing one quantum bivector for each face of the exploded complex. On this space there are operators $E_i^{pq}(T)$ $(1 \leq p, q \leq 4)$ generating one $\mathfrak{so}(4)$ action for each face $i = 0, 1, 2, 3$ of each tetrahedron T. Then we pick out states for which

$$\left(\hat{E}_0(T) + \hat{E}_1(T) + \hat{E}_2(T) + \hat{E}_3(T)\right) = 0$$

for all T in Δ. With some technical calculations this gives the subspace

$$\bigotimes_{T \in \delta_3} \mathcal{T}^+ \otimes \mathcal{T}^-$$

where

$$\mathcal{T} := \bigotimes_{j_0, j_1, j_2, j_3} \mathrm{Inv}(j_0 \otimes j_1 \otimes j_2 \otimes j_3)$$

where j_0, j_1, j_2, j_3 range over all possible labellings by spins and 'Inv' denote the subspace of vectors transforming under the trivial representation, is called the *Hilbert space of the quantum tetrahedron*. $\mathcal{T}^+$ and $\mathcal{T}^-$ are left-handed and right-handed copies of this Hilbert space.

If we define the left-handed and right-handed area operators by

$$\hat{A}_i^{\pm}(T) = \frac{1}{2}\sqrt{(\hat{E}_i^{\pm}(T) \cdot \hat{E}_i^{\pm}(T))}$$

then the closure constraint implies that these two operators must act in the same way, so we are lead to search a subspace of the previous Hilbert space of vectors satisfying that property.

It turns out that, if $\partial_0 T, \partial_1 T, \partial_2 T, \partial_3 T$ are the faces of a tetrahedron T in Δ and Δ_3 is the set of 3-simplices in Δ, this subspace is given by

$$\bigoplus_{\rho} \bigotimes_{T \in \Delta_3} \mathrm{Inv}(\rho_{\partial_0 T} \otimes \rho_{\partial_1 T} \otimes \rho_{\partial_2 T} \otimes \rho_{\partial_3 T}),$$

where ρ ranges over all labellings of triangles in Δ by representations of $Spin(4)$, or in other words, pairs of spins.

Finally we impose the quadratic constraints in the following quantized form:

$$(\hat{E}_i(T) \wedge \hat{E}_i(T))\psi = 0.$$

States satisfying these constraints form *Hilbert space of the quantum 4-simplex*. Barrett and Crane [24] explicitly describe this space with a restriction on the label ρ, i.e. they show that all the constraints are satisfied for the vectors of $L^2(\mathcal{A}_S/\mathcal{G}_S)$ lying in the subspace

$$\bigoplus_{\rho} \bigotimes_{T \in \Delta_3} (\mathrm{Inv}\rho_{\partial_0 T} \otimes \rho_{\partial_1 T} \otimes \rho_{\partial_2 T} \otimes \rho_{\partial_3 T}),$$

where ρ ranges only over labellings of triangles in Δ by representations of the form $j \otimes j$!

Thus the constraint 'left area=right area' imposes to label each triangle with a representation of the form $j \otimes j$ and each tetrahedron with an intertwiner of the form

$$I_j : V_{j_1} \otimes V_{j_2} \to V_{j_3} \otimes V_{j_4}$$

where V_{j_k} is the support space of the representation j_k and and $j_1, \ldots, j_4$ are the spins labelling the 4 triangular faces of the tetrahedron. More generally, we can label the tetrahedron by any intertwiner of the form $\sum_j c_j(I_j \otimes I_j)$ that satisfy the property 'left area=right area' for every splitting of the tetrahedra.

Barrett and Crane found an intertwiner with this property:

$$I = \sum_j (2j+1)(I_j \otimes I_j).$$

Later, Reisenberger [79] proved that this was the unique solution.

Hence, in the spin foam model proposed by Barrett and Crane, the transition amplitudes between a 4-geometry and another are given by a sum over spin foams that depends on spin and intertwiners of the form written above.

We conclude the chapter with some few remarks.

1. Bivectors E with $E \wedge E$ are called *simple* or *balanced*; these are precisely the bivectors that can be written as a wedge product of two vectors in $\mathbb{R}^4$. We may thus call $\bigoplus_j V_j \otimes V_j$ the *Hilbert space of a simple quantum bivectors*. By the Peter-Weyl theorem, this space is isomorphic to $L^2(SU(2))$;

2. Freidel, Krasnov and Puzio [39] have found an abstract way to impose the Barrett-Crane constraints that generalizes to the Lorentzian case. We will analyze this in the next chapter;

3. even if the Barrett Crane model gives a specific proposal for a spin foam model of quantum gravity, it relies on a *fixed* triangulation of spacetime. Researches to find out an upgrade of the model that doesn't depend on any fixed triangulation are under examination.

Chapter 7

The Barrett-Crane model for Lorentzian quantum gravity

7.1 Relativistic spin networks and Barrett-Crane model

In this chapter we will deal with the formulation of covariant quantum gravity with graphs instead of the dual picture presented in the previous chapter. The reason is that, recently, Freidel, Krasnov e Puzio showed in [39] that the constraints described before, applied on the labelling of the edges of the graphs are satisfied if one labels them with some special representations of G.

Precisely we define a 'relativistic spin network' to be a graph with edges labelled by non-negative real numbers that parameterize the class of unitary and irreducible representations of the connected component of the Lorentz group, i.e. $SO_0(3,1)$, or of its double cover $SL(2,\mathbb{C})$ that appear in the direct integral decomposition of $L^2(SL(2,\mathbb{C})/SU(2))$.

Starting from the fact that $SL(2,\mathbb{C})/SU(2)$ is a Riemannian symmetric spaces of non-compact type and using the Plancherel decompositions for these kind of spaces, due, among others, to Harish-Chandra and Helgason, we show that the Barrett-Crane labelling of the edges of relativistic spin networks can be thought as a special case of a more general situation in which $SL(2,\mathbb{C})$ is replaced by a non-compact semisimple Lie group of finite center G and $SU(2)$ is replaced with its maximal compact subgroup K.

7.2 Abstract results on the Plancherel decomposition of $L^2(G/K)$

A general reference for what follows is [76] page 195.

Let G be a non-compact semisimple Lie group with finite center, such a group is always unimodular, i.e. the left and the right Haar measure coincide. Let K be a maximal compact subgroup of G.

For simplicity let's denote the right coset space G/K by X and write with x its elements, that are of the form $gK := \{gk \mid k \in K\}$, where g is a fixed element of G.

There is a bijective correspondence between **right-K-invariant functions on** G and functions on X, i.e. those satisfying $f(gk) = f(g)$ for every $g \in G$ and $k \in K$.

In fact if $\mathcal{F}_K(G)$ is any space of right-K-invariant functions on G, then we can define the corresponding functional space on X by:

$$\mathcal{F}(X) := \{f \in \mathcal{F}_K(G) \mid f(gK) = f(g),\ g \in G\}.$$

Conversely, from a function on X we can get a function on G with the help of the natural projection of G onto X:

$$\begin{array}{rccc} \pi : & G & \longrightarrow & X \\ & g & \mapsto & \pi(g) := gK, \end{array}$$

in fact, given $f : X \to \mathbb{C}$, we can compose f with π to get

$$\begin{array}{rccc} f \circ \pi : & G & \longrightarrow & \mathbb{C} \\ & g & \mapsto & f(\pi(g)) \equiv f(gK). \end{array}$$

An important fact is that on the quotient space X there is a natural G-invariant measure[1] induced from the Haar measures of G and K. This measure can be defined by means of a theorem (see [49] page 91) which assures that the linear map

$$\begin{array}{ccc} \mathcal{C}_c(G) & \longrightarrow & \mathcal{C}_c(X) \\ f & \mapsto & \tilde{f}, \end{array}$$

$\tilde{f}(x) := \int_K f(gk)dk$ is *onto*[2]. Then it can be proved that X has a positive measure dx unique up to scalar multiples and defined by:

$$\int_X \tilde{f}(x)dx \equiv \int_{G/K}\int_K f(gK)dk\, d(gK) := \int_G f(g)dg \qquad f \in \mathcal{C}_c(G).$$

[1]A G-invariant measure μ on X is a measure satisfying $\int_X f(gx)d\mu = \int_X f(x)d\mu$ for every $g \in G$.

[2]The function $\tilde{f}$ is well defined on X, in fact, for every $k' \in K$, $\tilde{f}(gk') = \int_K f(gk'k)d(k'k)$, but substituting $\tilde{k} \equiv k'k$ and using the invariance of the Haar measure on K we get $\tilde{f}(gk') = \int_K f(g\tilde{k})d\tilde{k} = \int_K f(gk)dk = \tilde{f}(gk)$.

Since $L^2(G)$ is the completion of $\mathcal{C}_c(G)$ in the norm $\|f\|_2 := (\int_G |f(g)|^2 dg)^{\frac{1}{2}}$ and since $\|\tilde{f}\|_2 \leq \|f\|_2$, for every $f \in L^2(G)$, the projection $f \twoheadrightarrow \tilde{f}$ extends (see [67] page 226) from $\mathcal{C}_c(G)$ to $L^2(G)$ and its image is precisely $L^2(X)$.

By identifying the Hilbert spaces $L^2(X)$ and $L^2(G)$ with their dual spaces and taking the transposed map of the previous projection we obtain a one-to-one injection from $L^2(X)$ to $L^2(G)$.

Remember now that the left regular representation L of G on $L^2(G)$, i.e.

$$\begin{array}{rccc} L: & G & \longrightarrow & \mathcal{U}(L^2(G)) \\ & g & \mapsto & L(g), \end{array}$$

$L(g)f(h) := f(gh)$, $f \in L^2(G)$, $h \in G$, is a unitary representation that can be decomposed in direct integral as

$$L \simeq \int_{\hat{G}}^{\oplus} m_\lambda \lambda \, d\mu(\lambda)$$

where $d\mu$ is the Plancherel measure and m_λ is the multiplicity of $\lambda \in \hat{G}$.

The corresponding decomposition of $L^2(G)$ is:

$$L^2(G) \simeq \int_{\hat{G}}^{\oplus} \mathcal{B}_{HS}(\mathcal{H}_\lambda) \, d\mu(\lambda)$$

where $\mathcal{H}_\lambda$ is the representation space of λ and $\mathcal{B}_{HS}(\mathcal{H}_\lambda)$ is the (Hilbert) space of Hilbert-Schmidt operators on $\mathcal{H}_\lambda$.

Let's now see what happens when we consider the left regular representation L_X of G on $L^2(X)$, i.e.

$$L_X(g)f(x) := f(gx) \qquad f \in L^2(X),\ g \in G, x \in X.$$

This is again a unitary representation of G thanks to the G-invariance of dx and thus it admits a direct integral decomposition. It can be proved that in this case only a particular class of representations can appear in this decomposition. This class is given by the so-called **K-spherical representations** $(\lambda, \mathcal{H}_\lambda^K)$ of G, i.e. those representations in $\hat{G}$ for which the space $\mathcal{H}_\lambda^K$ of the K-fixed vectors[3] in $\mathcal{H}_\lambda$ is *not trivial*.

It's known that, when the previous hypothesis on G and K are satisfied, then $\mathcal{H}_\lambda^K$ is a 1-dimensional subspace of $\mathcal{H}_\lambda$ for every λ, thus it is itself a Hilbert space, and the multiplicity m_λ is 1 for every K-spherical representation λ.

[3]i.e., explicitly, $\mathcal{H}_\lambda^K := \{x_\lambda \in \mathcal{H}_\lambda \mid \rho_\lambda(k)x_\lambda = x_\lambda \ \forall k \in K\}$, where $\rho_\lambda \in \lambda$.

If $\hat{G}_K$ is the subset of $\hat{G}$ given by the equivalence classes of K-spherical representations, then *the Plancherel measure $d\mu$ appearing in the direct integral decomposition of $L^2(G/K)$ has support on $\hat{G}_K$.*

Finally define the **Fourier transform** of a function $f \in \mathcal{C}_c^\infty(X)$ as the function $\hat{f}$ on $\hat{G}_K$ defined by

$$\hat{f}(\lambda) := \int_X f(x)\rho_\lambda^{-1}(x)dx$$

where ρ_λ is an arbitrary representative of the class $\lambda \in \hat{G}_K$. It can be proved that

$$\hat{f}(\lambda) \in \mathcal{H}_\lambda \otimes (\mathcal{H}_\lambda^K)^* \simeq \mathcal{H}_\lambda \otimes \mathbb{C} \simeq \mathcal{H}_\lambda.$$

Then we have the Plancherel theorem for $L^2(X)$.

Theorem 7.2.1 *The Fourier transform $f \mapsto \hat{f}$ extends to a unitary equivalence between $L^2(X)$ and $\int_{\hat{G}_K}^{\oplus} \mathcal{H}_\lambda d\mu(\lambda)$:*

$$L^2(G/K) \simeq \int_{\hat{G}_K}^{\oplus} \mathcal{H}_\lambda \, d\mu(\lambda).$$

Moreover

$$L \simeq \int_{\hat{G}_K}^{\oplus} \lambda \, d\mu(\lambda).$$

The basic problems in making harmonic analysis on G/K concrete are two: the explicit description of $\hat{G}_K$ and the determination of the Plancherel measure $d\mu(\lambda)$.

Both these problems have been beautifully solved by the works of Harish-Chandra and Helgason. They managed to parameterize the elements of $\hat{G}_K$ with certain roots of the Lie algebra of G and reduced the Plancherel measure to an Euclidean measure! To describe their work we need to remind some notations and results from abstract harmonic analysis on semisimple Lie groups.

7.3 Tools from harmonic analysis on Riemannian symmetric spaces

A very brief and clear reference for this section is [50] page 102.

Let's describe the objects we are interested in.

- G is a non-compact connected semisimple Lie group with finite center and with Lie algebra $\mathfrak{g}$;

- The Killing form $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$,
$$B(X,Y) := Tr(ad\,X\,ad\,Y)$$
(where $ad\,X : \mathfrak{g} \to \mathfrak{g}$, $(ad\,X)(Y) := [X,Y]$) is a symmetric bilinear non-degenerate form on $\mathfrak{g}$;

- A **Cartan involution** $\theta : \mathfrak{g} \to \mathfrak{g}$ of $\mathfrak{g}$ is an involutive (i.e. its square is $id_{\mathfrak{g}}$) automorphism of $\mathfrak{g}$ such that the symmetric bilinear form given by $B_\theta(X,Y) := -B(X,\theta Y)$ is positive definite on $\mathfrak{g} \times \mathfrak{g}$. A typical example of Cartan involution is $\theta(X) = -X^\dagger$, this holds for many Lie algebras, e.g. $\mathfrak{su}(n)$, $\mathfrak{so}(n)$, $\mathfrak{sp}(n)$, $\mathfrak{sl}(n,\mathbb{C})$, $\mathfrak{sl}(n,\mathbb{R})$, $\mathfrak{so}(n,\mathbb{C})$, $\mathfrak{so}(p,q)$, $\mathfrak{su}(p,q)$, $n,p,q \geq 1$;

- If $\mathfrak{k}$ and $\mathfrak{p}$ are the eigenspaces of θ relative to the eigenvalues $+1$ and -1, resp., then $\mathfrak{g}$ can be decomposed into direct sum $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, so that
$$\begin{array}{rccc} \theta : & \mathfrak{k} \oplus \mathfrak{p} & \longrightarrow & \mathfrak{k} \oplus \mathfrak{p} \\ & T + P & \mapsto & T - P \end{array}$$
and thence $\mathfrak{k}$ is the space of fixed points of θ. If Θ denotes the involutive automorphism of G such that $d_e\Theta = \theta$, then the subgroup of G defined by $K := \exp(\mathfrak{k})$ agrees with the set of fixed points of Θ and it can be proved to be a maximal compact subgroup of G. If the Cartan involution on $\mathfrak{g}$ is $\theta(X) = -X^\dagger$, then $\Theta(g) = (g^\dagger)^{-1}$;

- G/K has a unique analytic manifold structure such that the map
$$\begin{array}{ccc} \mathfrak{p} & \longrightarrow & G/K \\ X & \mapsto & \exp(X)K \end{array}$$
is a diffeomorphism;

- Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal Abelian Lie algebra contained in $\mathfrak{p}$ and define $A = \exp(\mathfrak{a})$. Denote with $\mathfrak{a}^*$ the real algebraic dual of $\mathfrak{a}$, i.e. $\mathfrak{a}^* := Hom_{\mathbb{R}}(\mathfrak{a},\mathbb{R})$ and with $\mathfrak{a}^*_{\mathbb{C}}$ its complexification, namely the space of complex-valued $\mathbb{R}$-linear functionals on $\mathfrak{a}$, i.e. $\mathfrak{a}^*_{\mathbb{C}} := Hom_{\mathbb{R}}(\mathfrak{a},\mathbb{C})$;

- A (restricted) **root** of $\mathfrak{g}$ w.r.t. $\mathfrak{a}$ is a linear functional $\alpha \in \mathfrak{a}^*$ for which there exists $X \in \mathfrak{g} \setminus \{0\}$ such that $[H,X] = \alpha(H)X$, $\forall H \in \mathfrak{a}$, i.e. the bracket between X and H is proportional to X by a constant which depends linearly by $H \in \mathfrak{a}$. For every root α we put
$$\mathfrak{g}_\alpha := \{X \in \mathfrak{g} \mid [H,X] = \alpha(H)X,\ \forall H \in \mathfrak{a}\}$$
this space is called the **root space** relative to α and its dimension $m_\alpha := dim(\mathfrak{g}_\alpha)$ is called its **multiplicity**;

- Σ is the set of the roots of $\mathfrak{g}$ and Σ^+ is the set of positive roots w.r.t. some given ordering;

- $\mathfrak{a}'$ is the open subset of $\mathfrak{a}$ where all the roots are $\neq 0$, the connected components of $\mathfrak{a}'$ are called the **Weyl chambers**. We fix the Weyl chamber $\mathfrak{a}^+$ to be:

$$\mathfrak{a}^+ := \{H \in \mathfrak{a} \mid \alpha(H) > 0 \;\forall \alpha \in \Sigma^+\}.$$

  $\mathfrak{a}^*_+$ is the subset of $\mathfrak{a}^*$ given by the dual elements of $\mathfrak{a}^+$, $A^+ := \exp \mathfrak{a}^+$ and $\overline{A^+}$ is its completion in A;

- Define the Lie algebra $\mathfrak{n} \simeq \bigoplus_{\alpha\in\Sigma^+} \mathfrak{g}_\alpha$ and write N for its Lie group. These are a nilpotent Lie algebra and Lie group, respectively;

- With these objects we can write two classical decompositions of G, called, respectively the **Cartan polar decomposition** and the **Iwasawa decompositions**:

$$G \simeq K\overline{A^+}K, \qquad G \simeq KAN \simeq NAK$$

  i.e. the map $K \times \overline{A^+} \times K \ni (k_1, a(g), k_2) \mapsto k_1 a(g) k_2 \in G$ is a diffeomorphism, and the same property holds for the Iwasawa decompositions. The maps

$$\begin{array}{cccc} A: & G & \to & \mathfrak{a}^+ \\ & nak & \mapsto & A(g) := \log(a) \end{array} \qquad \begin{array}{cccc} H: & G & \to & \mathfrak{a}^+ \\ & k'a'n' & \mapsto & H(g) := \log(a') \end{array}$$

  (where the symbol log obviously denotes the inverse map of exp) are called **Iwasawa projections**. In terms of them every $g \in G$ can be written in a unique way as

$$g = ne^{A(g)}k \in NAK,$$

$$g = k'e^{H(g)}n' \in KAN.$$

  The relation between the two Iwasawa projections is: $A(g) = -H(g^{-1})$;

- It can be proved that the restriction of the Killing form B to $\mathfrak{a} \times \mathfrak{a}$ is a symmetric bilinear positive-definite non-degenerate form. This implies that we can use B as an inner product on $\mathfrak{a}$ to write

$$< \alpha, \beta >:= B(H_\alpha, H_\beta),$$

  where H_α is the only element of $\mathfrak{a}$ such that $\alpha(H) = B(H, H_\alpha)$, for every $H \in \mathfrak{a}$, and the same for H_β;

- ρ is the half sum of the positive roots counted with their multiplicity, i.e. $\rho := \frac{1}{2} \sum_{\alpha \in \Sigma^+} m_\alpha \alpha$;

- M is the centralizer of A in K, i.e. the compact subgroup of K containing the elements which commute with every element of A:

$$M := \{k \in K \mid kak^{-1} = a, \forall a \in A\};$$

- M' is the normalizer of A in K, i.e.

$$M' := \{k \in K \mid kak^{-1} \in A, \forall a \in A\};$$

- W is the Weyl group w.r.t. Σ, i.e. the finite group $W := M'/M$ generated by the reflections of the roots. W acts on $\mathfrak{a}^*$ and $\mathfrak{a}^*_{\mathbb{C}}$ by $(w\lambda)(H) = \lambda(w^{-1}H)$ for $H \in \mathfrak{a}$, $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ or $\lambda \in \mathfrak{a}^*$ and $w \in W$;

- The element $a(g)$ appearing in the Cartan polar decomposition is unique if we require it to be in $\overline{A^+}$, but if we ask it to be just in A then it is unique up to a Weyl transformation;

- Finally, the map

$$\begin{array}{ccc} K/M \times A^+ & \longrightarrow & G/K \\ (kM, a) & \mapsto & (ka)K \end{array}$$

  is a diffeomorphism onto a dense subset $X_+ \subset G/K$.

Let's now introduce the concept of **symmetric space**. A symmetric space is a connected Riemannian manifold $\mathcal{M}$ such that at each point $p \in \mathcal{M}$ there is an isometry[4] $s_p : \mathcal{M} \to \mathcal{M}$ that upsets the geodesic passing through p, i.e., if γ is a geodesic such that $\gamma(0) = p$, then $s_p(\gamma(t)) = \gamma(-t)$, $\forall t$.

It can be proved that $\mathcal{M}$ is a symmetric space if and only if for every point $p \in \mathcal{M}$ there is an involutive isometry s_p different from the identity which has p as the only fixed point.

The quotient G/K can be seen as a symmetric space in this way [89] page 269:

- call $\pi : G \to G/K$ the canonical map $\pi(g) := gK$;

- introduce a Riemannian metric Q on G/K by translating the Killing form on the space $\mathfrak{p}$:

$$Q_{gK}((d_g\pi)\tilde{X}_g, (d_g\pi)\tilde{Y}_g) \qquad \forall X, Y \in \mathfrak{p}$$

[4] i.e. , indicated with d the Riemannian metric of $\mathcal{M}$, $d(s_p(x), s_p(y)) = d(x, y)$ for every $x, y \in \mathcal{M}$

where $\tilde{X}$ denotes the left invariant vector field corresponding to $X \in \mathfrak{p}$. The metric Q is well defined because the Killing form is invariant under Ad_k, $k \in K$. Moreover it is obvious that the metric is positive by definition of $\mathfrak{p}$ and it's also easy to see that is G-invariant;

- consider the involutive automorphism Θ on G defined by $d_e\Theta = \theta$, where θ is a Cartan involution;
- then the geodesic-reversing isometry s_O at the origin O, which is obviously the coset K in G/K, is obtained from the involutive automorphism Θ as follows:

$$\begin{array}{rccc} s_O: & G/K & \longrightarrow & G/K \\ & gK & \mapsto & \Theta(g)K; \end{array}$$

- finally s_O can be translated by elements of G to obtain geodesic-reversing isometries in every point of G/K.

An important class of functions on G are the bi-invariant functions on G, i.e. those satisfying $f(kgk') = f(g)$, $\forall k, k' \in K, g \in G$ or the complex-valued functions on $K \backslash G/K$.

In the set of bi-invariant functions there are some particular functions which play a fundamental role in the harmonic analysis on G/K, these are the so-called **spherical functions** on G. A function $f \in \mathcal{C}^\infty(G)$ is said to be spherical if

1. f is bi-invariant;
2. $f(e_G) = 1$;
3. f is an eigenfunction of *each* differential operator $D \in \mathbf{D}_K(G)$, where $\mathbf{D}_K(G)$ is the algebra of all left-invariant differential operators on G which are also right-invariant under K, i.e. $Df = c_D f$, with $c_D \in \mathbb{C}$, for every $D \in \mathbf{D}_\mathbf{K}(G)$.

The spherical functions on G are in one-to-one correspondence with the ones on G/K, which are defined as functions $f \in \mathcal{C}^\infty(G/K)$ satisfying:

1. $f((kg)K) = f(gK)$ for every $k \in K$;
2. $f(e_G K) = 1$;
3. f is an eigenfunction of *each* differential operator $D \in \mathbf{D}(G/K)$, where $\mathbf{D}(G/K)$ is the algebra of G-left invariant operators on G/K.

There is a simple *characterization of spherical functions by means of an integral equation*, in fact it can be proved ([49] page 400) that $f \in \mathcal{C}(G), f \neq \mathbf{0}$, is spherical if and only if

$$\int_K f(gkg')dk = f(g)f(g') \qquad g, g' \in G, k \in K$$

where the integral is taken w.r.t. the normalized Haar measure dk on K.

The explicit realization of the spherical functions on G is possible thanks to a theorem due to Harish-Chandra ([49] page 418).

Theorem 7.3.1 *As λ varies in $\mathfrak{a}^*_{\mathbb{C}}$, the functions*

$$\phi_\lambda(g) := \int_K e^{(i\lambda+\rho)(A(kg))}\, dk \qquad g \in G,$$

exhaust the class of the spherical functions on G. Moreover $\phi_\lambda = \phi_{\lambda'}$ if and only if $\lambda' = w\lambda$, for some $w \in W$, i.e. if and only if λ and λ' are related by a transformation of the Weyl group.

In the statement of the theorem $A(kg)$ denotes the Iwasawa projection associated to the element $kg \in G$, but of course we can use the other Iwasawa projection, namely H, and remembering that $A(g) = -H(g^{-1})$ we have $\phi_\lambda(g) = \int_K e^{-(i\lambda+\rho)(H(g^{-1}k))}\, dk$, for every $g \in G$. But it is also true (same reference as before) that

$$\phi_{-\lambda}(g) = \phi_\lambda(g^{-1}) \qquad \forall g \in G,\ \lambda \in \mathfrak{a}^*_{\mathbb{C}}$$

thus the spherical functions can be written in two equivalent ways as:

$$\phi_\lambda(g) = \int_K e^{(i\lambda+\rho)(A(kg))}\, dk = \int_K e^{(i\lambda-\rho)(H(gk))}\, dk \qquad g \in G, \lambda \in \mathfrak{a}^*_{\mathbb{C}}.$$

7.3.1 Helgason decomposition of $L^2(G/K)$

The reference is [93] page 342 and [50] page 131.

For simplicity let's denote with B the compact quotient K/M and with db its K-invariant measure.

Given $\lambda \in \mathfrak{a}^*$ let $\mathcal{H}_\lambda$ denote the vector space

$$\mathcal{H}_\lambda := \{\phi_\lambda : X \to \mathbb{C} \mid \phi_\lambda(x) := \int_B f(b)e^{(i\lambda-\rho)(H(gk))}\, db, \text{ with } f \in L^2(B)\}.$$

It can be shown that the correspondence $L^2(B) \ni f \to \varphi_\lambda \in \mathcal{H}_\lambda$ is one-to-one and so $\mathcal{H}_\lambda$ inherits the structure of Hilbert space from $L^2(B)$ if we endow it with the inner product

$$(\phi_\lambda|\psi_\lambda) := \int_B \overline{f}(b)h(b)\,db$$

where obviously f and h are the functions in $L^2(B)$ uniquely associated to φ_λ and ψ_λ. Hence each Hilbert space $\mathcal{H}_\lambda$ is isomorphic to $L^2(B)$. It can be also proved that the left regular representations of G on $\mathcal{H}_\lambda$, i.e.

$$\begin{array}{rccl} L_\lambda: & G & \longrightarrow & \mathcal{U}(\mathcal{H}_\lambda) \\ & g & \mapsto & L_\lambda(g),\ L_\lambda(g)\phi_\lambda(x) := \phi_\lambda(g^{-1}x) \end{array}$$

are unitary and irreducible for every $\lambda \in \mathfrak{a}^*$ and constitute the building blocks to decompose the left regular representation on G on $L^2(X)$, and thus $L^2(X)$ itself, as proved by Helgason in the following theorem.

Theorem 7.3.2 *Let L_X denote the left regular representation of G on $L^2(X)$. Then $\{L_\lambda\}_{\lambda\in\mathfrak{a}^*}$ is equivalent to the set of K-spherical representations of the unitary principal series of G. Also there exists a function $\mathbf{c} : \mathfrak{a}^* \to \mathbb{C}$, called the* **Harish-Chandra c function***, whose modulus is constant on the conjugation classes of $\mathfrak{a}^*$ w.r.t. W such that the following direct integral decompositions hold:*

$$L_X \simeq \int_{\mathfrak{a}^*_+}^{\oplus} L_\lambda |\mathbf{c}(\lambda)|^{-2}\,d\lambda;$$

$$L^2(X) \simeq \int_{\mathfrak{a}^*_+}^{\oplus} \mathcal{H}_\lambda |\mathbf{c}(\lambda)|^{-2}\,d\lambda.$$

In the case of a complex group G it can be proved that

$$\mathbf{c}(\lambda) = \frac{\prod_{\alpha\in\Sigma^+} < \alpha, \rho >}{\prod_{\alpha\in\Sigma^+} < \alpha, i\lambda >}.$$

More generally, for semisimple Lie groups the $\mathbf{c}$-function can be written in terms of the Gindikin-Karpelevic formula (see [49]).

7.4 Derivation of the Barrett-Crane model for a 4-dimensional spacetime

References: [49] pages 432-433 and [89] page 311.

In this section we show explicit realizations of the object discussed previously for the case $G \equiv SL(2,\mathbb{C})$ and $K \equiv SU(2)$. This will show how the Barrett-Crane labelling of the edges of relativistic spin networks can be derived from the structure of Riemannian symmetric space of $SL(2,\mathbb{C})/SU(2)$.

First of all the quotient $SL(2,\mathbb{C})/SU(2)$ is the *three-dimensional real hyperbolic space* $\mathbf{H^3}$ and it's a symmetric space of type IV in the Cartan classification these kind of spaces.

The Iwasawa decomposition of $SL(2,\mathbb{C})$ is realized with:

- $K = SU(2)$;
- $A = \left\{ A_t \equiv \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \mid t \in \mathbb{R} \right\}$;
- $N = \left\{ \begin{pmatrix} 1 & x+iy \\ 0 & 1 \end{pmatrix} \mid x, y \in \mathbb{R} \right\}$.

A is a subgroup of the Cartan subgroup H of $SL(2,\mathbb{C})$, which is

$$H = \left\{ \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} \mid z \in \mathbb{C} \right\}.$$

H is the image, via the exponential map, of the Cartan subalgebra

$$\mathfrak{h} = \left\{ \begin{pmatrix} \zeta & 0 \\ 0 & -\zeta \end{pmatrix} \mid \zeta \in \mathbb{C} \right\}.$$

The subalgebra $\mathfrak{a} \subset \mathfrak{h}$ is

$$\mathfrak{a} = \left\{ a_t \equiv \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix} \mid t \in \mathbb{R} \right\}.$$

The centralizer of A in K is

$$M = \left\{ \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix}, \, t \in \mathbb{R} \right\},$$

this can be easily proved by considering the Cayley-Klein parametrization of the matrices of $SU(2)$: any such matrix can be written as $\begin{pmatrix} \beta & \gamma \\ -\bar{\gamma} & \bar{\beta} \end{pmatrix}$, with $\beta, \gamma \in \mathbb{C}$ satisfying the constraint $|\beta|^2 + |\gamma|^2 = 1$; then the matrices of $SU(2)$ commute with all those of A if and only if the matrix

$$\begin{pmatrix} \beta & \gamma \\ -\bar{\gamma} & \bar{\beta} \end{pmatrix} \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} = \begin{pmatrix} \beta e^t & \gamma e^{-t} \\ -\bar{\gamma} e^t & \bar{\beta} e^{-t} \end{pmatrix}$$

equals the matrix

$$\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \begin{pmatrix} \beta & \gamma \\ -\bar{\gamma} & \bar{\beta} \end{pmatrix} = \begin{pmatrix} \beta e^t & \gamma e^t \\ -\bar{\gamma} e^{-t} & \bar{\beta} e^{-t} \end{pmatrix}$$

for every $t \in \mathbb{R}$, but this is possible if and only if $\gamma = 0$ and, taking into account the constraint $|\beta|^2 + |\gamma|^2 = 1$, β must be of the form e^{it} for $t \in \mathbb{R}$, which proves that M has the form written above.

For $SL(2, \mathbb{C})$, Σ^+ consists only of the single root α given by $\alpha(a_t) = 2t$ and its multiplicity is $m_\alpha = 2$.

Hence $\mathfrak{a}^*$ is a 1-dimensional real vector space and $\mathfrak{a}^*_{\mathbb{C}}$ is a 1-dimensional complex vector space, thus they can be identified with $\mathbb{R}$ and $\mathbb{C}$, respectively:

$$\mathfrak{a}^* \simeq \mathbb{R}, \qquad \mathfrak{a}^*_{\mathbb{C}} \simeq \mathbb{C}$$

also,

$$\mathfrak{a}^*_+ \simeq \mathbb{R}^+ .$$

Let's write $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ in the form $\lambda \equiv l\alpha$, with $l \in \mathbb{C}$. Since the spherical functions are bi-K-invariant, and thanks to the polar Cartan decomposition, the spherical functions on $SL(2, \mathbb{C})$ are determined by their values on the matrices of A, that, for this purpose, can be more conveniently expressed as $\begin{pmatrix} e^{\frac{t}{2}} & 0 \\ 0 & e^{-\frac{t}{2}} \end{pmatrix}$, then

$$\phi_{l\alpha} \begin{pmatrix} e^{\frac{t}{2}} & 0 \\ 0 & e^{-\frac{t}{2}} \end{pmatrix} = \frac{\sin(lt)}{l \sinh(t)}.$$

It also follows that the root $\rho := \frac{1}{2} \sum_{\alpha \in \Sigma^+} m_\alpha \alpha$ reduces simply to $\frac{1}{2} \cdot 2\alpha = \alpha$ and so the Harish-Chandra **c**-function can be written as:

$$\mathbf{c}(\lambda) = \frac{< \alpha, \alpha >}{< \alpha, il\alpha >} = -\frac{i}{l},$$

hence $|\mathbf{c}(\lambda)|^{-2} = l^2$ and the Harish-Chandra measure is $\mathbf{c}(\lambda) d\lambda = l^2 \, dl$.

Finally the Weyl group of $SL(2, \mathbb{C})$ is the two-elements group Z_2 and the action of the non trivial element takes takes $\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}$ into $\begin{pmatrix} z^{-1} & 0 \\ 0 & z \end{pmatrix}$.

We are now able to write down the integral decomposition of the Hilbert space $L^2(SL(2, \mathbb{C})/SU(2))$, in fact putting together all the explicit computations above we get that, if $p \in \mathbb{R}^+$, then

$$L^2(SL(2, \mathbb{C})/SU(2)) \simeq \int_{\mathbb{R}^+}^{\oplus} \mathcal{H}_p \, p^2 \, dp.$$

Since the representations satisfying the quantum version of the constraints appearing in the Barrett-Crane state sum model are the $SU(2)$-spherical representations (see e.g. [39]) and since these representations are precisely the ones that appear in the direct integral decomposition of $L^2(SL(2,\mathbb{C})/SU(2))$ (as remembered at the beginning of this paper), our discussion shows that the labelling of the Barrett-Crane relativistic spin networks with positive real numbers is a natural consequence of the intrinsic analytic structure of the Riemannian symmetric space $SL(2,\mathbb{C})/SU(2)$.

However our construction holds for more general groups G and K than $SL(2,\mathbb{C})$ and $SU(2)$, respectively, and this could be used to propose an extension of the Barrett-Crane construction of relativistic spin networks to higher dimensions. We will give a proposal for an abstract model in the next section.

Finally notice that the integral kernel of the evaluations of the relativistic spin networks (see [20]) are nothing but the spherical functions labelled with p, i.e.

$$K_p(r) = \frac{\sin(pr)}{p \sinh(r)}$$

where r is the hyperbolic distance between two points in $\mathbf{H^3}$.

The behavior of the spherical functions is well known thanks to the asymptotic expansion formulas of Harish-Chandra and others [49], this could be used to evaluate the relativistic spin networks for more general groups.

7.5 $SL(n,\mathbb{C})$-Barrett-Crane relativistic spin networks

In the previous section we have shown how to derive the Barrett-Crane labelling of the edges of the 4-dimensional relativistic spin networks from the structure of G/K, where $G = SL(2,\mathbb{C})$ and $K = SU(2)$, in fact, with these choices, the set $\mathfrak{a}_+^*$ identifies with $\mathbb{R}^+$ and one has the direct integral decomposition of $L^2(SL(2,\mathbb{C})/SU(2))$ written before.

By analogy, we give here an account of what happens if the gauge group of the model is $G = SL(n,\mathbb{C})$ and K is $SU(n)$, $n > 2$. This is only an abstract model, since the 'true' n-dimensional Barrett-Crane model is related to the group $SO_0(n-1,1)$, but the harmonic analysis on such a group is too difficult to handle here because the Plancherel decomposition admits both a continuous and a discrete part, and this latter is not yet fully known!

The Iwasawa decomposition of $SL(n,\mathbb{C})$ is realized with:

- $K = SU(n)$;

- $A = \{A_{\underline{t}} \equiv \mathrm{diag}(e^{t_1}, \ldots, e^{t_n}) \mid t_1, \ldots, t_n \in \mathbb{R},\ \sum_{i=1}^n t_i = 0\}$;
- N is given by the $n \times n$ upper triangular matrices with complex entries and with all 1 on the diagonal.

The algebra $\mathfrak{a}$ is

$$\mathfrak{a} = \{a_{\underline{t}} \equiv \mathrm{diag}(t_1, \ldots, t_n) \mid t_1, \ldots, t_n \in \mathbb{R}, \sum_{i=1}^{n} t_i = 0\}.$$

The relation $\sum_{i=1}^n t_i = 0$ gives a constraint so that $dim_{\mathbb{R}}(\mathfrak{a}) = dim_{\mathbb{R}}(\mathfrak{a}^*) = n-1$, $dim_{\mathbb{C}}(\mathfrak{a}^*_{\mathbb{C}}) = n-1$, hence, as vector spaces $\mathfrak{a} \simeq \mathfrak{a}^* \simeq \mathbb{R}^{n-1}$, $\mathfrak{a}^*_{\mathbb{C}} \simeq \mathbb{C}^{n-1}$.

The restricted roots of $SL(n, \mathbb{C})$ can be constructed by means of the linear functionals $e_i \in \mathfrak{a}^*$ defined by

$$e_i(\mathrm{diag}(t_1, \ldots, t_n)) := t_i$$

in fact notice that, if E_{ij} is the elementary matrix with 1 in the position (i, j) and 0 in all the other entries, then

$$(ad\, a_{\underline{t}})E_{ij} \equiv [a_{\underline{t}}, E_{ij}] = \mathrm{diag}(t_1, \ldots, t_n)E_{ij} - E_{ij}\mathrm{diag}(t_1, \ldots, t_n) = (t_i - t_j)E_{ij}$$

but $t_i - t_j = (e_i - e_j)a_{\underline{t}}$ and so $[a_{\underline{t}}, E_{ij}] = (e_i - e_j)(a_{\underline{t}})E_{ij}$. It follows that the restricted root system is

$$\Sigma = \{e_i - e_j,\ i, j = 1, \ldots, n, i \neq j\}$$

the condition $i \neq j$ guarantees that all the roots in Σ are different from zero.

With respect to the natural lexicographic ordering the *positive restricted roots* are given by

$$\Sigma^+ = \{e_i - e_j,\ i < j\}.$$

Now we want to find a simple characterization of the set

$$\mathfrak{a}^+ = \{a_{\underline{t}} \in \mathfrak{a} \mid (e_i - e_j)(a_{\underline{t}}) > 0\ \forall\, i < j\}.$$

By iteratively applying the roots $e_i - e_j$, $i < j$, to $a_{\underline{t}}$ one obtains the condition $t_i > t_j$ for every $i < j$, which implies $t_1 > t_2 > \ldots > t_{n-1} > t_n$, moreover the traceless condition on $a_{\underline{t}}$ implies that $t_n = -\sum_{i=1}^{n-1} t_i$, which is compatible with the previous one if and only if $t_1, t_2, \ldots, t_{n-1}$ are all strictly greater than zero, hence

$$\mathfrak{a}^+ = \left\{ \mathrm{diag}(t_1, \ldots, t_n) \mid t_1 > t_2 > \ldots > t_{n-1} > 0, t_n = -\sum_{i=1}^{n-1} t_i \right\},$$

but then the elements of $\mathfrak{a}^+$ are uniquely individuated by a strictly decreasing string of $n-1$ positive real numbers. The same thing can be said for the elements of $\mathfrak{a}^*_+$ because they are in one-to-one correspondence with those of $\mathfrak{a}^+$ thanks to the bijection induced by the Killing form.

This result is important for the purpose of the Plancherel decomposition of $L^2(SL(n,\mathbb{C})/SU(n))$, since the direct integral that realizes this decomposition is performed over $\mathfrak{a}^*_+$ and what shown above implies that this set can be more conveniently represented as

$$\mathcal{P}_n \equiv (P_1, +\infty) \times (0, P_2) \times \cdots \times (0, P_{n-1}),$$

with $P_1, P_2, \ldots, P_{n-1} \in \mathbb{R}$, $P_1 > P_2 > \ldots > P_{n-1} > 0$.

An explicit basis of $\mathfrak{a}^*$ is given by $(\delta_1, \ldots, \delta_{n-1})$, where $\delta_k := e_k - e_{k+1}$, $k = 1, \ldots, n-1$, hence the generic element $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ can be written as $\lambda = \sum_{k=1}^{n-1} l_k \delta_k, l_k \in \mathbb{C}$. This follows from the fact that the roots δ_k are $n-1$ *simple*[5] roots, which are known to be linearly independent.

Since $\rho = \frac{1}{2}\sum_{i<j} 2(e_i - e_j) = \sum_{i<j} e_i - e_j$, the Harish-Chandra **c**-function is given by[6]

$$\mathbf{c}(\lambda) = -\iota \frac{\prod_{i<j} < e_i - e_j, \sum_{i<j} e_i - e_j >}{\prod_{i<j} < e_i - e_j, \sum_{k=1}^{n-1} l_k \delta_k >}$$

if λ is written as $\sum_{k=1}^{n-1} l_k \delta_k$, $l_k \in \mathbb{C}$, $k = 1, \ldots, n-1$.

We now have all the elements to write down the Plancherel decomposition of $L^2(SL(n,\mathbb{C})/SU(n))$ as:

$$\int_{\mathcal{P}_n}^{\oplus} \mathcal{H}_{(p_1,\ldots,p_{n-1})} \left(\frac{\prod_{i<j} < e_i - e_j, \sum_{k=1}^{n-1} p_k \delta_k >}{\prod_{i<j} < e_i - e_j, \sum_{i<j} e_i - e_j >} \right)^2 dp_1 \cdots dp_{n-1},$$

$(p_1, \ldots, p_{n-1}) \in \mathcal{P}_n$.

This implies that *the labelling of the edges of the relativistic spin network for an abstract Barrett-Crane model with gauge group $SL(n,\mathbb{C})$ is given by associating $(n-1)$ distinct positive real numbers to every edge*[7].

The inner products $< \cdot, \cdot >$ can be easily computed thanks to the formula

$$< e_i - e_j, e_r - e_s >= B(H_{ij}, H_{rs})$$

[5]A root α is called simple if it is positive and it doesn't decompose as $\alpha = \beta_1 + \beta_2$, with β_1 and β_2 both positive roots.

[6]To avoid confusion we write the imaginary unit, i.e. $\sqrt{-1}$, with ι.

[7]In fact these ones can be suitably permutated to form a strictly decreasing string of positive real numbers.

where the element $H_{ij} \in \mathfrak{a}$, uniquely associated to the restricted root $e_i - e_j$, is given by the matrix

$$H_{ij} = \frac{1}{2n}(E_{ii} - E_{jj})$$

and the same obviously holds for H_{rs}. Moreover the Killing form is given by

$$B(X,Y) = 2n\,Tr(XY).$$

7.5.1 Explicit computations for n=3

To have an explicit example at hand we analyze the case of $n = 3$, i.e. the symmetric space $SL(3,\mathbb{C})/SU(3)$.

To calculate the Plancherel measure we have to find out the Harish-Chandra **c**-function:

$$\mathbf{c}(p_1,p_2) = -\iota \frac{\prod_{i<j} < e_i - e_j, \sum_{i<j} e_i - e_j >}{\prod_{i<j} < e_i - e_j, \sum_{k=1,2} p_k\delta_k >}$$

with $i,j = 1,2,3$ and $p_1,p_2 \in \mathcal{P}_3$.

The only thing that we have to do is to calculate the inner products that appear in the previous formula. To avoid cumbersome notations let's put $e_i - e_j \equiv e_{ij}$.

At the numerator we have $\prod_{i<j} < e_{ij}, \rho >= (1)\cdot(2)\cdot(3)$ where

$$(1) =< e_{12}, e_{12} > + < e_{12}, e_{13} > + < e_{12}, e_{23} >= \frac{1}{3} + \frac{1}{6} - \frac{1}{6};$$

$$(2) =< e_{13}, e_{12} > + < e_{13}, e_{13} > + < e_{13}, e_{23} >= \frac{1}{6} + \frac{1}{3} + \frac{1}{6};$$

$$(3) =< e_{23}, e_{12} > + < e_{23}, e_{13} > + < e_{23}, e_{23} >= -\frac{1}{6} + \frac{1}{6} + \frac{1}{3};$$

hence $\prod_{i<j} < e_{ij}, \rho >= \frac{2}{27}$.

At the denominator instead we have $\prod_{i<j} < e_{ij}, \sum_{k=1,2} p_k\delta_k >= (I)\cdot(II)\cdot(III)$ where

$$(I) =< e_{12}, p_1 e_{12} > + < e_{12}, p_2 e_{23} >= \frac{1}{3}p_1 - \frac{1}{6}p_2;$$

$$(II) =< e_{13}, p_1 e_{12} > + < e_{13}, p_2 e_{23} >= \frac{1}{6}p_1 + \frac{1}{6}p_2;$$

$$(III) =< e_{23}, p_1 e_{12} > + < e_{23}, p_2 e_{23} >= -\frac{1}{6}p_1 + \frac{1}{3}p_2;$$

thus $\prod_{i<j} < e_{ij}, \sum_{k=1,2} p_k\delta_k >= \frac{1}{54}(p_1 - \frac{1}{2}p_2)(p_1 + p_2)(p_2 - \frac{1}{2}p_1)$.

Combining the calculations we find that the Plancherel decomposition for $n = 3$ is:

$$L^2(SL(3,\mathbb{C})/SU(3)) \simeq 16 \int_{\mathcal{P}_3}^{\oplus} \mathcal{H}_{(p_1,p_2)} \left[\left(p_1 - \frac{1}{2}p_2\right)(p_1 + p_2)\left(p_2 - \frac{1}{2}p_1\right)\right]^2 dp_1\, dp_2.$$

7.6 Barrett-Crane intertwiners for relativistic spin networks

7.6.1 The Euclidean case

In their article [38], Freidel and Krasnov show how to write the Barrett-Crane intertwiners for the n-dimensional Euclidean quantum gravity − corresponding to the gauge group $SO(n)$ − as integrals over the homogenous space $SO(n-1)\backslash SO(n)$, which can be identified with the $(n-1)$-dimensional sphere $\mathbb{S}^{n-1}$.

Their construction basically relies on a personal interpretation of the Peter-Weyl theorem.

First of all they showed in [39] the key fact that the representations ρ_e of $SO(n)$ that label the edges of the relativistic spin networks appearing in the Barrett-Crane model are those for which the representation spaces V^{ρ_e} contain non-trivial vectors invariant under the action of $SO(n-1)$. They call these representations '*class-1 representations*' of $SO(n)$, but they are better known in the books of harmonic analysis as '$SO(n-1)$*-spherical representations*' and in the literature of quantum gravity as '*simple (or balanced) representations*'.

More generally we can talk about K-spherical representations of the compact semisimple group G, where K is a maximal closed (hence compact) subgroup of G.

The collection of all K-spherical representations of G defines a subset of the dual object $\hat{G}$, indicated with $\hat{G}_K$ and called the *K-spherical dual of G*.

One of the most important features of the K-spherical representations is that the matrix elements of the unitary matrices they define can be written as integrals over the homogeneous space $X \equiv K\backslash G$, let's see how this can be done.

By using the notation and results of chapter 2, we remember here that the *matrix coefficient maps* $\rho_{ij} : G \to \mathbb{C}$, $g \mapsto \rho_{ij}(g)$, are square-integrable on G, moreover, fixed a row index $i \equiv \bar{\imath}$, the vector space $\mathfrak{M}_\rho$ generated by

$\rho_{\bar{\imath}j}$, i.e.

$$\mathfrak{M}_\rho := \{\sum_{j=1}^{d(\rho)} c_j \rho_{\bar{\imath}j},\ c_j \in \mathbb{C}\ \forall j = 1, \ldots, d(\rho)\}$$

can be proved to be a R-invariant subspace of $L^2(G)$, where R is the right-regular representation of G. The Peter-Weyl theorem can be stated in terms of the following direct sum decompositions of $L^2(G)$ and R:

$$L^2(G) \simeq \bigoplus_{\rho \in \hat{G}} d(\rho) \mathfrak{M}_\rho;$$

$$R \simeq \bigoplus_{\rho \in \hat{G}} d(\rho) \rho.$$

The right regular representation is not irreducible, but it can be restricted to a subrepresentation R_ρ on $\mathfrak{M}_\rho$ which is irreducible:

$$\begin{array}{rccc} R_\rho: & G & \longrightarrow & \mathcal{U}(\mathfrak{M}_\rho) \\ & g & \mapsto & R_\rho, \end{array}$$

$R_\rho(g) := R(g)|_{\mathfrak{M}_\rho}, \forall g \in G$.

This representation turns out to be equivalent to ρ, i.e. there exists a unitary intertwiner I such that the following diagram commutes:

$$\begin{array}{ccc} V^\rho & \xrightarrow{I} & \mathfrak{M}_\rho \\ {\scriptstyle \rho(g)}\downarrow & & \downarrow {\scriptstyle R_\rho(g)} \\ V^\rho & \xrightarrow[I]{} & \mathfrak{M}_\rho \end{array}$$

for all $g \in G$.

Notice that this unitary equivalence enables to identify $\rho(g)$ with a shift operator $R_\rho(g)$ on $\mathfrak{M}_\rho$, thus one can calculate the matrix elements of $\rho(g)$ dealing with the more convenient operator $R_\rho(g)$. In fact, if $\{\tilde{u}_n^\rho\}(n \in \mathbb{N})$ is an orthonormal basis of $\mathfrak{M}_\rho$, then

$$\rho_{nm}(g) = (R_\rho(g)\tilde{u}_m^\rho | \tilde{u}_n^\rho) = \int_G \overline{\tilde{u}_m^\rho(hg)} \tilde{u}_n^\rho(h) dh.$$

Thanks to the R-invariance of the spaces $\mathfrak{M}_\rho$, the integral in the formula above projects down to $X = K\backslash G$ and we reach the formula that expresses the matrix elements of $\rho(g)$ as integrals over X:

$$\rho_{nm}(g) = \int_X \overline{u_m^\rho(xg)} u_n^\rho(x) dx$$

where, u^ρ_n denotes the projection of $\tilde{u}^\rho_n$ on X with the help of the canonical map $\pi : G \to K\backslash G$, $g \mapsto Kg$: $\tilde{u}^\rho_n =: u^\rho_n \circ \pi$.

Comparing the evaluation of a spin network written with these integral expressions for the matrix elements with the evaluation expressed in the usual way, Freidel and Krasnov were able to derive an integral formula valid for the intertwiners that label the vertices of the spin networks under analysis.

If v is a k-valent vertex, with $e_1, \dots, e_i$ edges having *source* in v and $e_{i+1}, \dots, e_k$ having *target* in v, then a general intertwiner labelling v is a linear operator

$$\iota_v : V^{\rho_{e_{i+1}}} \otimes \cdots \otimes V^{\rho_{e_k}} \longrightarrow V^{\rho_{e_1}} \otimes \cdots \otimes V^{\rho_{e_i}}$$

or, equivalently, a mixed tensor with components $(\iota_v)^{n_{i+1}\dots n_k}_{m_1\dots m_i}$.

Since every V^{ρ_j}, $j = 1, \dots, k$, is isomorphic to a space of the type $\mathfrak{M}_{\rho_j}$, the right-translation invariance of these spaces enables to write the components of the intertwiner as integrals over $X = K\backslash G$:

$$(\iota_v)^{n_{i+1}\dots n_k}_{m_1\dots m_i} = \int_X (\overline{u^{\rho_1}_{m_1}(x)} \cdots \overline{u^{\rho_i}_{m_i}(x)}) \cdot (u^{\rho_{i+1}}(x)^{n_{i+1}} \cdots u^{\rho_k}(x)^{n_k})\, dx.$$

7.6.2 The Lorentzian case

Now let's examine the case of Lorentzian gravity in 4-dimensions. The groups G and K of the previous section have to be replaced by $SL(2,\mathbb{C})$ and $SU(2)$, respectively. We know that the quotient space X is isomorphic to the three-dimensional real hyperbolic space.

Due to the non-compactness of the groups in question, the decomposition of $L^2(X)$ is not given by a direct sum anymore, but by a direct integral, specifically:

$$L^2(X) \simeq \int^{\oplus}_{\mathfrak{a}^*_+} \mathcal{H}_\lambda\, |\mathbf{c}(\lambda)|^{-2} d\lambda,$$

where $\mathfrak{a}^*_+ \simeq \mathbb{R}^+$, $\mathbf{c}(\lambda)$ is the Harish-Chandra $\mathbf{c}$-function, that in this case is given by $\mathbf{c}(p) = \frac{1}{p}$, $p \in \mathbb{R}^+$, and the Hilbert spaces $\mathcal{H}_p$ are defined by

$$\mathcal{H}_p := \{\varphi_p : X \to \mathbb{C} \mid \varphi_p(x) := \int_B f(b) e^{(i\lambda-\rho)(H(gk))}\, db, \text{ with } f \in L^2(B)\}.$$

Remember that $H : G \to \mathfrak{a}^+$ denotes the Iwasawa projection, ρ is the half sum of the positive roots weighted with their multiplicities and $B := K/M$, where M is the centralizer of A in K, being A the maximal Abelian subgroup of G appearing its Iwasawa decomposition.

M can be proved to be of the form

$$M = \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix},\ \theta \in [0, 2\pi] \right\},$$

thus it is isomorphic to the torus group $\mathbb{T} \simeq U(1)$ and so

$$B \simeq SU(2)/U(1).$$

Moreover every Hilbert space $\mathcal{H}_p$ is isomorphic to $L^2(B)$ and so $L^2(X)$ decomposes in fact in the direct integral of the *constant* field of Hilbert spaces defined by: $\mathbb{R}^+ \ni p \mapsto \mathcal{H}_p \simeq L^2(B)$. Moreover the left and right regular representations decompose in direct integral as well with decompositions:

$$L \simeq \int_{\mathbb{R}^+}^{\oplus} L_p \, p^2 \, dp$$

where

$$\begin{array}{rccl} L_p : & G & \longrightarrow & \mathcal{U}(\mathcal{H}_p) \\ & g & \mapsto & L_p(g),\ L_p(g)\varphi_p(x) := \varphi_p(g^{-1}x) \end{array}$$

and similar for R.

Thus the integral formula of Freidel and Krasnov that defines the intertwiner of a k-valent vertex seems to be generalizable to the present case by an operator-valued integral of a suitable product of elements of an orthonormal basis of $\mathcal{H}_{p_j} \simeq L^2(B)$, namely:

$$(\iota_v)^{n_{i+1}\dots n_k}_{m_1 \dots m_i} = \int_X (\overline{u^{p_1}_{m_1}(x)} \cdots \overline{u^{p_i}_{m_i}(x)}) \cdot (u^{p_{i+1}}(x)^{n_{i+1}} \cdots u^{p_k}(x)^{n_k}) \, dx.$$

The big difference with the compact case is that here the u's can be identified with elements of an orthonormal basis of a single Hilbert space: $L^2(B)$. The reason is that here the representation spaces of the K-spherical irreps that come into play are all infinite-dimensional separable Hilbert space and they admit a sort of 'standard copy', i.e. $L^2(B)$.

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