On the almost sure invariance principle for stationary sequences of Hilbert-valued random variables

Dedicated to the memory of Walter Philipp

J. Dedecker*and F. Merlevède†

Abstract

We prove an almost sure invariance principle for the partial sums of a strictly stationary sequence of Hilbert-valued random variables. As a consequence, we extend the almost sure invariance principle of Dehling and Philipp (1982) for strongly mixing sequences, by giving the balance between the strong mixing rate and the moment of the norm of the variables, for any moment greater than 2. We also show that our result holds for many non mixing sequences, including a class of Hilbert-valued Markov chain which appears when studying randomly forced partial differential equations.

Mathematics Subject Classifications (2000): 60 F 15.
Key words: weakly dependent sequences, Hilbert space, maximal inequalities, almost sure invariance principle.
Short Title: Almost invariance principle in Hilbert spaces.

1 Introduction and Notations

Let us start with few notations.

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, and $T : \Omega \mapsto \Omega$ be a bijective bi-measurable transformation preserving the probability $\mathbb{P}$. We denote by $\mathcal{I}$ the σ-algebra of T-invariant sets. The map T is $\mathbb{P}$-ergodic if each element of $\mathcal{I}$ has measure 0 or 1. Let $\mathcal{F}_0$ be a sub-σ-algebra of $\mathcal{A}$ satisfying $\mathcal{F}_0 \subseteq T^{-1}(\mathcal{F}_0)$.

*Université Paris 6-Pierre et Marie Curie, Laboratoire de Statistique Théorique et Appliquée.
†Université Paris Est-Marne-la-Vallée, LAMA and CNRS UMR 8050.

Let $\mathbb{H}$ be a real separable Hilbert space with norm $\|\cdot\|_{\mathbb{H}}$ generated by an inner product $<\cdot,\cdot>$. Let X_0 be a $\mathcal{F}_0$-measurable $\mathbb{H}$-valued random variable, such that $\mathbb{E}(\|X_0\|_{\mathbb{H}}^2) < \infty$ and $\mathbb{E}(X_0) = 0_{\mathbb{H}}$. Define then the stationary sequence $(X_i)_{i\in\mathbb{Z}}$ by $X_i = X_0 \circ T^i$, and the partial sums $S_n = X_1 + \cdots + X_n$.

In this paper, we shall give sufficient conditions under which the almost sure invariance principle (ASIP) holds: there exists a sequence $(Z_i)_{i\in\mathbb{Z}}$ of iid $\mathbb{H}$-valued gaussian random variables with $\mathbb{E}(\|Z_0\|_{\mathbb{H}}^2) < \infty$ and $\mathbb{E}(Z_0) = 0_{\mathbb{H}}$, and such that

$$\Big\| S_n - \sum_{i=1}^{n} Z_i \Big\|_{\mathbb{H}} = o(\sqrt{n \ln(\ln(n))}) \quad \text{almost surely.} \tag{1.1}$$

Let us briefly describe the application of our main result (Theorem 4) to the case of strongly mixing sequences in the sense of Rosenblatt (1956). Recall that the strong mixing coefficient between two σ-algebras $\mathcal{F}$ and $\mathcal{G}$ is defined by

$$\alpha(\mathcal{F},\mathcal{G}) = \sup_{A\in\mathcal{F}, B\in\mathcal{G}} |\mathbb{P}(A\cap B) - \mathbb{P}(A)\mathbb{P}(B)|\,.$$

For the sequence $(X_i)_{i\in\mathbb{Z}}$ and the σ-algebra $\mathcal{F}_0 = \sigma(X_i, i \le 0)$, define then

$$\alpha_X(n) = \alpha(\mathcal{F}_0, \sigma(X_i, i \ge n)) \quad \text{and} \quad \alpha_{2,X}(n) = \sup_{i\ge j\ge n} \alpha(\mathcal{F}_0, \sigma(X_i, X_j))\,. \tag{1.2}$$

Note that, by definition, $\alpha_{2,X}(n) \le \alpha_X(n)$.

Corollary 1. *Let Q be the cadlag inverse of the tail function $t \to \mathbb{P}(\|X_0\|_{\mathbb{H}} > t)$. If*

$$\sum_{k=1}^{\infty} \int_0^{\alpha_{2,X}(k)} Q^2(u)du < \infty\,, \tag{1.3}$$

then, enlarging Ω if necessary, there exists a sequence $(Z_i)_{i\in\mathbb{Z}}$ of iid $\mathbb{H}$-valued gaussian random variables with $\mathbb{E}(\|Z_0\|_{\mathbb{H}}^2) < \infty$ and $\mathbb{E}(Z_0) = 0_{\mathbb{H}}$, and such that (1.1) holds. The covariance operator of Z_0 is given by (2.8).

If $\mathbb{H} = \mathbb{R}$, the condition (1.3) is weaker than Rio's condition (1995) for real-valued strongly mixing sequences :

$$\sum_{k=1}^{\infty} \int_0^{\alpha_X(k)} Q^2(u)du < \infty\,. \tag{1.4}$$

We refer to Theorem 3 in Rio's paper for a discussion about the optimality of (1.4).

In 2008, Merlevède proved that the compact law of the iterated logarithm holds for Hilbert-valued random variables under the condition (1.4). Concerning the ASIP for strongly mixing sequences, the best result known until now was that by Dehling and Philipp (1982), who obtained the ASIP (1.1) under the condition: for $0 < \delta \leq 1$ and $\varepsilon > 0$,

$$\mathbb{E}(\|X_0\|_{\mathbb{H}}^{2+\delta}) < \infty \quad \text{and} \quad \alpha_X(n) = O(n^{-(1+\varepsilon)(1+2/\delta)})\,. \tag{1.5}$$

Many new tools were developed by Dehling and Philipp in this seminal paper. Among them, let us cite a covariance inequality for bounded strongly mixing Hilbert-valued random variables (see their Lemma 2.2) which is based on a deep result from Banach space theory given in Lindenstrauss and Tzafriri (1977).

To compare the condition of Dehling and Phillip (1982) with (1.3), note that (1.3) holds as soon as

$$\mathbb{E}(\|X_0\|_{\mathbb{H}}^{2+\delta}) < \infty \quad \text{for some } \delta > 0 \text{ and} \quad \sum_{k=1}^{\infty} k^{2/\delta}\alpha_{2,X}(k) < \infty\,,$$

which is weaker than (1.5), and is true without any limitation on δ. In particular, if $\|X_0\|_{\mathbb{H}} \leq M$ almost surely, we obtain the ASIP under the simple condition $\sum_{k>0} \alpha_{2,X}(k) < \infty$.

In fact, we shall prove in Theorem 4 a much more general result than Corollary 1. The condition is expressed in terms of a dependence coefficient which is much weaker than $\alpha_{2,X}$ and can be computed for many non strongly mixing processes. In particular, our result also improves on Theorem 3 in Dedecker and Merlevède (2006) for τ-dependent sequences (see Section 4.2). In the bounded case (Theorem 2) our conditions are even weaker, and are expressed in terms of conditional expectations, in the spirit of Corollary 2(δ) in Dedecker and Merlevède (2003) for the central limit theorem.

The starting point of this paper is the remark made by Merlevède (2008), that no coupling techniques can lead to the result given in Corollary 3, because of a counter example given by Dehling (1983), and that new tools have to be developed. She then proved a new Fuk-Nagaev inequality by using a blocking technique and a martingale approximation of the blocks. As a consequence, she obtained first a bounded law of the iterated logarithm with a precise upper bound, which enables her to go back to the d-dimensional case, and to obtain the compact law of the iterated logarithm.

However, at this stage, an essential argument was missing to obtain the ASIP: find under the same assumption as for the compact law, a suitable martingale approximation of $< S_n, x >$ in order to use the equivalence between the ASIP and the compact law given in Theorem 3.2 of Berger (1990). This point will be fully explained in Section 5.3.

We shall also prove a new Fuk-Nagaev inequality (Theorem 3), which improves on Theorem 1 in Merlevède (2008). The proof is more complicated than in Merlevède's paper, for we only control the dependence between the past σ-algebra $\mathcal{F}_0$ and two points X_i, X_j through lipschitz functions of the norms $\|X_i + X_j\|_{\mathbb{H}}$ and $\|X_i - X_j\|_{\mathbb{H}}$.

2 ASIP: the bounded case

Our first result is a maximal inequality for partial sums of bounded random variables.

Theorem 1. *Assume that $\|X_0\|_{\mathbb{H}} \le M$ almost surely for a $M > 0$. Let*

$$\delta(n) = \max\Big\{\mathbb{E}(\|\mathbb{E}(X_n|\mathcal{F}_0)\|_{\mathbb{H}}), \sup_{i\ge j\ge n} \frac{1}{M}\|\mathbb{E}(< X_i, X_j > |\mathcal{F}_0) - \mathbb{E}(< X_i, X_j >)\|_1\Big\}.$$

For any $x > 0$, $r \ge 1$, and $s_n > 0$ with $s_n^2 \ge n \sum_{i=0}^{n-1} |\mathbb{E}(< X_0, X_i >)|$, one has

$$\mathbb{P}\Big(\sup_{1\le k\le n} \|S_k\|_{\mathbb{H}} \ge 4x\Big) \quad \le \quad 4\exp\left(-\frac{r^2 s_n^2}{2x^2} h\left(\frac{x^2}{r s_n^2}\right)\right) + n\Big\{\frac{1}{x} + \frac{x}{r s_n^2}\Big\}\delta([x/rM])\,,$$

where $h(u) := (1+u)\ln(1+u) - u$.

The next result is an almost sure invariance principle. We need the following definition.

Definition 1. A nonnegative self adjoint operator Γ on a separable Hilbert space $\mathbb{H}$ will be called an $\mathcal{S}(\mathbb{H})$-operator if it has finite trace, i.e. for some (and therefore every) orthonormal basis $(e_l)_{l\ge1}$ of $\mathbb{H}$, $\sum_{l\ge1} < \Gamma e_l, e_l > < \infty$.

Theorem 2. *Assume that $\|X_0\|_{\mathbb{H}} \le M$ almost surely for a $M > 0$. Assume that the two following conditions hold*

$$\sum_{n>0} \mathbb{E}\|\mathbb{E}(X_n|\mathcal{F}_0)\|_{\mathbb{H}} < \infty\,, \tag{2.6}$$

$$\sum_{n>0} \sup_{i\ge j\ge n} \|\mathbb{E}(< X_i, X_j > |\mathcal{F}_0) - \mathbb{E}(< X_i, X_j >)\|_1 < \infty\,. \tag{2.7}$$

Then

1. *The following control holds*

$$\sum_{n>0} \frac{1}{n}\mathbb{P}\Big(\sup_{k\in[1,n]} \|S_n\|_{\mathbb{H}} \ge A\sqrt{n\ln(\ln(n)))}\Big) < \infty\,,$$

   for $A = 8\sqrt{2}\big(\mathbb{E}(\|X_0\|_{\mathbb{H}}^2) + \sum_{k>0} |\mathbb{E}(< X_0, X_k >)|\big)^{1/2}$.

2. *The operator Γ defined for any x and y in $\mathbb{H}$ by*

$$< x, \Gamma y >= \sum_{k=0}^{\infty} \mathbb{E}(< X_0, x >< X_k, y >) + \sum_{k=1}^{\infty} \mathbb{E}(< X_0, y >< X_k, x >) \tag{2.8}$$

   is in $\mathcal{S}(\mathbb{H})$.

3. Assume moreover that T is $\mathbb{P}$-ergodic, and that, for any $x, y \in \mathbb{H}$,

$$\sum_{n>0} \sup_{i\geq j\geq n} \|\mathbb{E}(< X_i, x >< X_j, y > |\mathcal{F}_0) - \mathbb{E}(< X_i, x >< X_j, y >)\|_1 < \infty\,. \tag{2.9}$$

Then, enlarging Ω if necessary, there exists a sequence $(Z_i)_{i\in\mathbb{Z}}$ of iid $\mathbb{H}$ valued gaussian random variables with mean $0_{\mathbb{H}}$ and covariance operator Γ such that

$$\Big\|S_n - \sum_{i=1}^{n} Z_i\Big\|_{\mathbb{H}} = o(\sqrt{n\ln(\ln(n))}),\ \textit{almost surely}.$$

Remark 1. If $\mathbb{H} = \mathbb{R}$ and T is $\mathbb{P}$-ergodic, we obtain the ASIP for bounded random variables under the simple condition

$$\sum_{n>0} \|\mathbb{E}(X_n|\mathcal{F}_0)\|_1 < \infty \quad \text{and} \quad \sum_{n>0} \sup_{i\geq j\geq n} \|\mathbb{E}(X_iX_j|\mathcal{F}_0) - \mathbb{E}(X_iX_j)\|_1 < \infty\,.$$

This result is new, and is comparable to Gordin's $\mathbb{L}^1$-criterion (1973) in the bounded case: the central limit theorem holds as soon as $\sum_{n>0} \|\mathbb{E}(X_n|\mathcal{F}_0)\|_1 < \infty$.

3 ASIP: the general case

Before stating the results, we shall introduce some notations and definitions concerning the weak dependence coefficients used in this section.

Definition 2. For any integrable random variable Y, define the "upper tail" quantile function Q_Y by $Q_Y(u) = \inf\{t \geq 0 : \mathbb{P}(|Y| > t) \leq u\}$. Note that, on the set $[0, \mathbb{P}(|Y| > 0)]$, the function $H_Y : x \to \int_0^x Q_Y(u)du$ is an absolutely continuous and increasing function with values in $[0, \mathbb{E}|Y|]$. Denote by G_Y the inverse of H_Y.

Definition 3. Let $\Lambda_1(\mathbb{H})$ be the set of functions from $\mathbb{H}$ to $\mathbb{R}$ such that $|f(x)-f(y)| \leq \|x-y\|_{\mathbb{H}}$. For any σ-algebra $\mathcal{F}$ of $\mathcal{A}$ and any $\mathbb{H}$-valued integrable random variable X, we consider the coefficient $\theta(\mathcal{F}, X)$ defined by

$$\theta(\mathcal{F}, X) = \sup_{f\in\Lambda_1(\mathbb{H})} \|\mathbb{E}(f(X)|\mathcal{F}) - \mathbb{E}(f(X))\|_1\,. \tag{3.10}$$

We now define the coefficients $\gamma(n)$, $\theta_2(n)$ and $\lambda_2(n)$ of the sequence $(X_i)_{i\in\mathbb{Z}}$.

Definition 4. For any positive integer k, define

$$\theta_2(n) = \sup_{i\geq j\geq n} \max\{\theta(\mathcal{F}_0, X_i + X_j), \theta(\mathcal{F}_0, X_i - X_j)\} \quad \text{and} \quad \gamma(n) = \mathbb{E}(\|\mathbb{E}(X_n|\mathcal{F}_0)\|_{\mathbb{H}})\,. \tag{3.11}$$

Let now

$$\lambda_2(n) = \theta_2(n) \vee \gamma(n)\,. \tag{3.12}$$

Our first result is a maximal inequality for partial sums.

Theorem 3. *Let* $Q = Q_{\|X_0\|_{\mathbb{H}}}$, $H = H_{\|X_0\|_{\mathbb{H}}}$ *and* $G = G_{\|X_0\|_{\mathbb{H}}}$. *Let*

$$R(u) = (\min\{q \in \mathbb{N} : \lambda_2(q) \le H(u)\} \wedge n)Q(u) \text{ and } S(v) = R^{-1}(v) = \inf\{u \in [0,1] : R(u) \le v\}.$$

For any $x > 0$, $r \ge 1$, *and* $s_n > 0$ *with* $s_n^2 \ge n \sum_{i=0}^{n-1} \int_0^{\gamma(i)} Q \circ G(u)du$, *one has*

$$\mathbb{P}\Big(\sup_{1 \le k \le n} \|S_k\|_{\mathbb{H}} \ge 4x \Big) \le 4 \exp\left(-\frac{r^2 s_n^2}{2x^2} h\left(\frac{x^2}{r s_n^2} \right) \right) + n \Big\{ \frac{2}{x} + \frac{24x}{r s_n^2} \Big\} \int_0^{S(x/r)} Q(u)du\,,$$

where $h(u) := (1+u)\ln(1+u) - u$.

Remark 2. Since $h(u) \ge u \ln(1+u)/2$, under the notations and assumptions of the above theorem, we get that for any $x > 0$ and $r \ge 1$,

$$\mathbb{P}\Big(\sup_{1 \le k \le n} \|S_k\|_{\mathbb{H}} \ge 4x \Big) \le 4 \left(1 + \frac{x^2}{r s_n^2} \right)^{-r/4} + n \Big\{ \frac{2}{x} + \frac{24x}{r s_n^2} \Big\} \int_0^{S(x/r)} Q(u)du\,. \qquad (3.13)$$

The next result is an almost sure invariance principle.

Theorem 4. *Let* $Q = Q_{\|X_0\|_{\mathbb{H}}}$ *and* $G = G_{\|X_0\|_{\mathbb{H}}}$. *If*

$$\sum_{k>0} \int_0^{\lambda_2(k)} Q \circ G(u)du < \infty\,, \qquad (3.14)$$

then

$$\sum_{n>0} \frac{1}{n} \mathbb{P}\Big(\sup_{k \in [1,n]} \|S_k\|_{\mathbb{H}} \ge A\sqrt{n \ln(\ln(n)))} \Big) < \infty\,, \qquad (3.15)$$

with $A = 8\sqrt{2} \sum_{i=0}^{\infty} \int_0^{\gamma(i)} Q \circ G(u)du$, *and the operator* Γ *defined by (2.8) is in* $\mathcal{S}(\mathbb{H})$.

Assume moreover that T *is* $\mathbb{P}$*-ergodic. Then, enlarging* Ω *if necessary, there exists a sequence* $(Z_i)_{i\in\mathbb{Z}}$ *of iid* $\mathbb{H}$ *valued gaussian random variables with mean* $0_{\mathbb{H}}$ *and covariance operator* Γ *such that*

$$\Big\| S_n - \sum_{i=1}^n Z_i \Big\|_{\mathbb{H}} = o(\sqrt{n \ln(\ln(n)))}), \text{ almost surely}. \qquad (3.16)$$

Remark 3. Using the same arguments as to prove Corollary 2(α) in Dedecker and Merlevède (2003), we see that the condition (3.14) is true provided that (1.3) holds. This proves Corollary 1 given in the introduction.

Remark 4. If $\mathbb{H} = \mathbb{R}$, then we have $\lambda_2(k) = \theta_2(k)$ in (3.12). Hence, if T is $\mathbb{P}$-ergodic, we obtain the ASIP under the simple condition

$$\sum_{k>0} \int_0^{\theta_2(k)} Q \circ G(u) du < \infty \,. \tag{3.17}$$

This result is new and is weaker than Rio's result (1995), since (1.4) implies (3.17). Note that the coefficients $\theta_2(k)$ can be easily controlled for many non strongly mixing sequences: see Section 4.2 below.

4 Applications

4.1 Cramer-von Mises statistics

Let Y_0 be a real-valued random variable with distribution function F, and let μ be some probability measure on the real line. Let $Y_i = Y_0 \circ T^i$ and $X_i(t) = \mathbf{1}_{Y_i \le t} - F(t)$. On the space $\mathbb{H} = \mathbb{L}^2(\mu)$, the sequence $(X_i)_{i\in\mathbb{Z}}$ is a strictly stationary sequence of bounded random variables, so that the results of Theorem 2 apply. Hence, if T is $\mathbb{P}$-ergodic, we obtain the ASIP in $\mathbb{L}^2(\mu)$ for the empirical process

$$\{n(F_n(t) - F(t)), t \in \mathbb{R}\} \quad \text{where} \quad n(F_n(t) - F(t)) = \sum_{i=1}^n X_i(t) = \sum_{i=1}^n (\mathbf{1}_{Y_i \le t} - F(t)) \,.$$

In particular, we obtain the ASIP as soon as $\sum_{k>0} \alpha_{2,Y}(k) < \infty$, where $\alpha_{2,Y}$ is the coefficient defined by (1.2). But Theorem 2 also applies to non strongly mixing sequences. More precisely, define the random variables

$$\begin{aligned} b_i(t) &= \mathbb{E}(\mathbf{1}_{Y_i \le t} | \mathcal{F}_0) - F(t) \\ b_{i,j}(s,t) &= \mathbb{E}((\mathbf{1}_{Y_i \le s} - F(s))(\mathbf{1}_{Y_j \le t} - F(t)) | \mathcal{F}_0) - \mathbb{E}((\mathbf{1}_{Y_i \le s} - F(s))(\mathbf{1}_{Y_j \le t} - F(t))) \,. \end{aligned}$$

Clearly, the two conditions (2.6) and (2.7) will hold as soon as

$$\sum_{i>0} \mathbb{E}\Big(\int (b_i(t))^2 \mu(dt) \Big)^{1/2} < \infty \quad \text{and} \quad \sum_{n>0} \sup_{i \ge j \ge n} \mathbb{E}\Big| \int b_{i,j}(t,t) \mu(dt) \Big| < \infty \,.$$

and the condition (2.9) holds as soon as

$$\sum_{n>0} \sup_{i \ge j \ge n} \mathbb{E}\Big(\int (b_{i,j}(s,t))^2 \mu(ds)\mu(dt) \Big)^{1/2} < \infty \,.$$

As in Dedecker and Prieur (2007), define the coefficients $\beta_2(k)$ by:

$$\beta_2(k) = \max\Big\{\mathbb{E}\Big(\sup_{t\in\mathbb{R}}|b_k(t)|\Big), \sup_{i\geq j\geq k}\mathbb{E}\Big(\sup_{s,t\in\mathbb{R}}|b_{i,j}(s,t)|\Big)\Big\}.$$

Clearly the three conditions (2.6), (2.7) and (2.9) of Theorem 4 hold as soon as

$$\sum_{k>0}\beta_2(k) < \infty\,, \tag{4.18}$$

and consequently, the ASIP holds as soon as (4.18) is satisfied. Many examples of non strongly mixing processes for which $\beta_2(n)$ can be computed are given in Dedecker and Prieur (2007).

As a consequence of the ASIP, we obtain the following almost sure result for Cramer-von Mises statistics: if (4.18) holds, then

$$\limsup_{n\to\infty}\frac{n}{2\ln(\ln(n))}\int(F_n(t)-F(t))^2\mu(dt) = \sup_{\|x\|_{\mathbb{H}}\leq 1} < x, \Gamma x > \quad \text{almost surely,} \tag{4.19}$$

where Γ is defined in (2.8). From Dedecker and Merlevède (2003) we also know that

$$n\int(F_n(t)-F(t))^2\mu(dt) \quad \text{converges in distribution to} \quad \int(G(t))^2\mu(dt)\,, \tag{4.20}$$

where G is a gaussian random variable with values in $\mathbb{L}^2(\mu)$ and covariance operator Γ. Note that the asymptotic results (4.19) and (4.20) cannot be directly used for testing goodness-of-fit (even with the usual choice $\mu = dF$), because the law of the limiting distribution depends on the covariance terms $(\mathbb{E}(< X_0, x >< X_k, y >))_{k\geq 0}$ which appear in the definition of the operator Γ. Starting from the limiting result (4.20), an interesting and non-trivial problem is then to find a statistical procedure for testing goodness-of-fit, maybe by estimating some of the eigenvalues of the operator Γ.

4.2 τ-dependent sequences

Let Y_0 be a random variable with values in a separable Banach space $(\mathbb{B}, |\cdot|_{\mathbb{B}})$, such that $\mathbb{E}(|Y_0|_{\mathbb{B}}) < \infty$. Let $Y_i = Y_0 \circ T^i$. For any $\alpha \in]0,1]$, let $\Lambda_1(\mathbb{B}, \alpha)$ be the set of functions f from $\mathbb{B}$ to $\mathbb{R}$ such that

$$|f(x)-f(y)| \leq |x-y|_{\mathbb{B}}^{\alpha}\,.$$

Let $\Lambda_1(\mathbb{B}^2, \alpha)$ be the set of functions f from $\mathbb{B}^2$ to $\mathbb{R}$ such that

$$|f(x_1,y_1)-f(x_2,y_2)| \leq \frac{1}{2}|x_1-y_1|_{\mathbb{B}}^{\alpha} + \frac{1}{2}|x_2-y_2|_{\mathbb{B}}^{\alpha}\,.$$

Define the dependence coefficients $\tau_{1,\alpha}$ and $\tau_{2,\alpha}$ of the sequence $(Y_i)_{i\in\mathbb{Z}}$ by

$$\begin{aligned}
\tau_{1,\alpha}(k) &= \Big\| \sup_{f\in\Lambda_1(\mathbb{B},\alpha)} \Big|\mathbb{E}(f(Y_k)|\mathcal{F}_0) - \mathbb{E}(f(Y_k))\Big|\Big\|_1 , \\
\tau_{2,\alpha}(k) &= \max\Big\{\tau_{1,\alpha}(k), \sup_{i>j\geq k}\Big\| \sup_{f\in\Lambda_1(\mathbb{B}^2,\alpha)} \Big|\mathbb{E}(f(Y_i,Y_j)|\mathcal{F}_0) - \mathbb{E}(f(Y_i,Y_j))\Big|\Big\|_1\Big\} .
\end{aligned}$$

Starting from Theorem 4 one can prove the two following corollaries. Let c be any concave function from $\mathbb{R}^+$ to $\mathbb{R}^+$, with $c(0)=0$. Let $\mathcal{L}_c$ be the set of functions f from $\mathbb{B}$ to $\mathbb{H}$ such that

$$\|f(x)-f(y)\|_{\mathbb{H}} \leq Kc(|x-y|_{\mathbb{B}}), \quad \text{for some positive } K.$$

If $c(x)=x^\alpha$ for some $\alpha\in]0,1]$, then $\mathcal{L}_c$ is exactly the set H_α of α-Hölder functions from $\mathbb{B}$ to $\mathbb{H}$.

Corollary 2. *Let $f\in H_\alpha$, and let $X_k=f(Y_k)-\mathbb{E}(f(Y_k))$. Assume that X_0 is in $\mathbb{L}^2(\mathbb{H})$, and let $Q=Q_{\|X_0\|_{\mathbb{H}}}$, and $G=G_{\|X_0\|_{\mathbb{H}}}$. If*

$$\sum_{k>0}\int_0^{\tau_{2,\alpha}(k)} Q\circ G(u)du < \infty,$$

then the conclusion of Theorem 4 holds.

Corollary 3. *Let $f\in\mathcal{L}_c$, and let $X_k=f(Y_k)-\mathbb{E}(f(Y_k))$. Assume that X_0 is in $\mathbb{L}^2(\mathbb{H})$, and let $Q=Q_{\|X_0\|_{\mathbb{H}}}$, and $G=G_{\|X_0\|_{\mathbb{H}}}$. If*

$$\sum_{k>0}\int_0^{c(\tau_{2,1}(k))} Q\circ G(u)du < \infty,$$

then the conclusion of Theorem 4 holds.

Note that Corollary 3 improves on Theorem 3 in Dedecker and Merlevède (2006). Many examples of stationary processes for which the coefficient $\tau_{2,1}$ can be easily computed may be found in Dedecker and Merlevède (2006, Section 3).

Let us give another example here.

4.2.1 $\mathbb{H}$-valued Markov chains for randomly forced PDE's

We consider the class of $\mathbb{H}$-valued auto-regressive process described in Section 3 of the paper by Masmoudi and Young (2002):

$$Z_{n+1} = S(Z_n) + \eta_n , \tag{4.21}$$

where $(\eta_i)_{i\in\mathbb{Z}}$ is a sequence of iid $\mathbb{H}$-valued random variables with marginal distribution ν and independent of Z_0, and $S:\mathbb{H}\mapsto\mathbb{H}$. The map S and the measure ν are assumed to satisfy the assumptions (**P1**) – (**P4**) page 466 and (**C**) page 467 in Masmoudi and Young (2002). Under these assumptions there is an unique invariant probability measure μ, which is compactly supported.

Let then $(Y_i)_{i\geq 0}$ be the stationary solution of (4.21). The chain $(Y_i)_{i\geq 0}$ is $\tau_{2,1}$-dependent with an exponential decay of the coefficients. More precisely, following the proof of Theorem A(2) given pages 470-471 in Masmoudi and Young, one can see that if $f \in \Lambda_1(\mathbb{H}^2, 1)$ and $x \in \text{supp}(\mu)$, there exist $C > 0$ and $\rho < 1$ such that

$$\text{for any } i, j \geq n, \quad \Big|\mathbb{E}(f(Y_i, Y_j)|Y_0 = x) - \int \mathbb{E}(f(Y_i, Y_j)|Y_0 = x)\mu(dx)\Big| \leq C\rho^n ,$$

which implies that $\tau_{2,1}(n) \leq C\rho^n$.

Applying Corollary 3, we infer that the ASIP holds for $X_k = f(Y_k) - \mathbb{E}(f(Y_k))$ as soon as $f \in \mathcal{L}_c$, with $\sum_{k>0} c(\rho^k) < \infty$. This last condition on c is equivalent to $\int_0^1 t^{-1}c(t)dt < \infty$, and is satisfied as soon as $c(t) \leq D|\ln(t)|^{-\gamma}$ for some $D > 0$ and $\gamma > 1$. In particular, it holds for any Hölder function f from $\mathbb{H}$ to $\mathbb{H}$.

Such $\mathbb{H}$-valued auto-regressive processes appear when studying a class of dissipative partial differential equations (PDE's) perturbed by a random kick-force. More precisely, let S_t be the resolving operator (or semigroup) of a PDE with initial condition u_0, that is $S_t(u(0)) = u(t)$, where u is a solution of the (non random) PDE with initial condition $u(0)$. The Markov chain is defined by $Z_{k+1} = S_1(Z_k) + \eta_k$, where η_k is a random kick force, that is an iid sequences of $\mathbb{H}$-valued random variable of the form $\sum_{j=1}^{\infty} b_j\xi_{j,k}e_j$, where $\sum_{j>0} b_j^2 < \infty$ and $\xi_{j,k}$ is a real-valued random variable with a compactly supported density. In particular the resolving operators S_1 of the Navier-Stokes or Ginzburg-Landau equations are shown to satisfy the conditions (**P1**)–(**P3**) mentioned above (the condition (**P4**) is a condition on the distribution of η_0).

Note that exponential mixing for lipschitz functions of such Markov chains was proved independently by Kuskin, Piatniski and Shirikyan (2002), with applications to dissipative PDE's perturbed by a random kick-force. See also the paper by Shirikyan (2006) for the law of large numbers and the central limit theorem in the non-stationary case (i.e. starting from an arbitrary point $Z_0 = z$ in $\mathbb{H}$). In this paper (Shirikyan (2006)), it is also proved that such Markov chains are in general not strongly mixing (see the example 1.3 page 224-225).

5 Proofs

5.1 Proof of Theorem 3

Let $M > 0$. For $i \geq 0$ define the variables

$$X_i' = X_i \mathbb{1}_{\|X_i\|_{\mathbb{H}} \leq M} \quad \text{and} \quad X_i'' = X_i \mathbb{1}_{\|X_i\|_{\mathbb{H}} > M} .$$

Let $S'_n = \sum_{i=1}^n X'_i$ and $S''_n = \sum_{i=1}^n X''_i$. Let q be a positive integer and for $1 \leq i \leq [n/q]$, define the random variables $U'_i = S'_{iq} - S'_{iq-q}$ and $U''_i = S''_{iq} - S''_{iq-q}$. With these notations, the following decomposition is valid

$$\max_{1\leq k\leq n} \|S_k\|_{\mathbb{H}} \leq \max_{1\leq j\leq [n/q]} \| \sum_{i=1}^{j} U'_i \|_{\mathbb{H}} + qM + \sum_{k=1}^{n} \|X''_k\|_{\mathbb{H}} \,. \tag{5.1}$$

Setting now for all $i \geq 1$, $\mathcal{F}_i^U = \sigma(X_j, j \leq iq)$, we define a sequence $(\tilde{U}_i)_{i\geq 1}$ as follows: for all $i \geq 1$, $\tilde{U}_{2i-1} = U'_{2i-1} - \mathbb{E}(U'_{2i-1}|\mathcal{F}^U_{2(i-1)-1})$ and $\tilde{U}_{2i} = U'_{2i} - \mathbb{E}(U'_{2i}|\mathcal{F}^U_{2(i-1)})$. Notice that $(\tilde{U}_i)_{i\geq 1}$ is a sequence of martingale differences with respect to $(\mathcal{F}^U_i)$. Substituting the variables $\tilde{U}_i$ to the initial variables, in the inequality (5.1), we derive the following upper bound

$$\begin{aligned}\max_{1\leq k\leq n} \|S_k\|_{\mathbb{H}} \leq qM + \max_{2\leq 2j\leq [n/q]} \Big\|\sum_{i=1}^{j} \tilde{U}_{2i}\Big\|_{\mathbb{H}} + \max_{1\leq 2j-1\leq [n/q]} \Big\|\sum_{i=1}^{j} \tilde{U}_{2i-1}\Big\|_{\mathbb{H}} \\ + \sum_{i=1}^{[n/q]} \|U'_i - \tilde{U}_i\|_{\mathbb{H}} + \sum_{k=1}^{n} \|X''_k\|_{\mathbb{H}} \,.\end{aligned} \tag{5.2}$$

Since $\|U'_i\|_{\mathbb{H}} \leq qM$ almost surely, it follows that $\|\tilde{U}_i\|_{\mathbb{H}} \leq 2qM$ almost surely. Then applying Lemma 1 of the appendix with $y = 2s_n^2$, we derive that

$$\begin{aligned}\mathbb{P}\Big(\max_{2\leq 2j\leq [n/q]} \Big\|\sum_{i=1}^{j} \tilde{U}_{2i}\Big\|_{\mathbb{H}} \geq x\Big) &\leq 2\exp\Big(-\frac{s_n^2}{2(qM)^2} h\Big(\frac{xqM}{s_n^2}\Big)\Big) \\ &+\mathbb{P}\Big(\sum_{i=1}^{[[n/q]/2]} \mathbb{E}(\|\tilde{U}_{2i}\|^2_{\mathbb{H}}|\mathcal{F}^U_{2(i-1)}) \geq 2s_n^2\Big) \,.\end{aligned} \tag{5.3}$$

Now notice that

$$\begin{aligned}\mathbb{E}(\|\tilde{U}_{2i}\|^2_{\mathbb{H}}|\mathcal{F}^U_{2(i-1)}) &= \mathbb{E}(\|U'_{2i}\|^2_{\mathbb{H}}|\mathcal{F}^U_{2(i-1)}) - \|\mathbb{E}(U'_{2i}|\mathcal{F}^U_{2(i-1)})\|^2_{\mathbb{H}} \\ &\leq \mathbb{E}(\|U'_{2i}\|^2_{\mathbb{H}}|\mathcal{F}^U_{2(i-1)}) \\ &\leq \sum_{k=(2i-1)q+1}^{2iq} \sum_{j=(2i-1)q+1}^{2iq} \mathbb{E}(< X'_j, X'_k > |\mathcal{F}_{2(i-1)q}) \,,\end{aligned}$$

and that by Inequality (3.33) in Dedecker and Merlevède (2003), we have that

$$\begin{aligned}\sum_{i=1}^{[[n/q]/2]} \sum_{k=(2i-1)q+1}^{2iq} \Big(\mathbb{E}(< X'_k, X_k >) + 2\sum_{j=(2i-1)q+1}^{k-1} \mathbb{E}(< X'_j, X_k >)\Big) &\leq n\sum_{i=0}^{q-1} \int_0^{\gamma(i)} Q\circ G(u)du \\ &\leq s_n^2 \,.\end{aligned}$$

Hence

$$\sum_{i=1}^{[[n/q]/2]} \mathbb{E}(\|\tilde{U}_{2i}\|_{\mathbb{H}}^2 | \mathcal{F}^U_{2(i-1)}) \le s_n^2 + \sum_{i=1}^{[[n/q]/2]} A_{1,i} + \sum_{i=1}^{[[n/q]/2]} A_{2,i} \,,$$

where

$$\begin{aligned} A_{1,i} &= \sum_{k=(2i-1)q+1}^{2iq} \sum_{j=(2i-1)q+1}^{2iq} \Big(\mathbb{E}(< X'_j, X'_k > | \mathcal{F}_{2(i-1)q}) - \mathbb{E}(< X'_j, X'_k >) \Big) \\ A_{2,i} &= \Big| \sum_{k=(2i-1)q+1}^{2iq} \Big(\mathbb{E}(< X'_k, X'_k - X_k >) + 2 \sum_{j=(2i-1)q+1}^{k-1} \mathbb{E}(< X'_j, X'_k - X_k >) \Big) \Big| \,. \end{aligned}$$

Applying Markov inequality, and noting that

$$\Big| \mathbb{E}\big(< X'_j, X'_k - X_k > \big) \Big| \le M \mathbb{E}(\|X_0\|_{\mathbb{H}} \mathbb{1}_{\|X_0\|_{\mathbb{H}} > M}) \,,$$

we infer that

$$\begin{aligned} \mathbb{P}\Big(\sum_{i=1}^{[[n/q]/2]} \mathbb{E}(\|\tilde{U}_{2i}\|_{\mathbb{H}}^2 | \mathcal{F}^U_{2(i-1)}) \ge 2 s_n^2 \Big) &\le \frac{nq}{2s_n^2} \sup_{j \ge k \ge q} \| \mathbb{E}(< X'_j, X'_k > | \mathcal{F}_0) - \mathbb{E}(< X'_j, X'_k >) \|_1 \\ &\quad + \frac{nq}{2s_n^2} M \mathbb{E}(\|X_0\|_{\mathbb{H}} \mathbb{1}_{\|X_0\|_{\mathbb{H}} > M}) \,. \end{aligned}$$

Notice now that for any $x \in \mathbb{H}$ and $y \in \mathbb{H}$,

$$4 < x, y >= \|x+y\|_{\mathbb{H}}^2 - \|x-y\|_{\mathbb{H}}^2 \,.$$

Also for any $x \in \mathbb{H}$ and $y \in \mathbb{H}$,

$$\begin{aligned} & \Big| \|x \mathbb{1}_{\|x\|_{\mathbb{H}} \le M} + y \mathbb{1}_{\|y\|_{\mathbb{H}} \le M}\|_{\mathbb{H}}^2 - \|x+y\|_{\mathbb{H}}^2 \mathbb{1}_{\|x+y\|_{\mathbb{H}} \le 2M} \Big| \\ & \le \|y\|^2 \mathbb{1}_{\|y\|_{\mathbb{H}} \le M} \mathbb{1}_{\|x\|_{\mathbb{H}} > M} + \|x\|^2 \mathbb{1}_{\|x\|_{\mathbb{H}} \le M} \mathbb{1}_{\|y\|_{\mathbb{H}} > M} + \|x+y\|^2 \mathbb{1}_{\|x+y\|_{\mathbb{H}} \le 2M} \mathbb{1}_{\|x\|_{\mathbb{H}} > M} \\ & \quad + \|x+y\|^2 \mathbb{1}_{\|x+y\|_{\mathbb{H}} \le 2M} \mathbb{1}_{\|y\|_{\mathbb{H}} > M} + \|x+y\|^2 \mathbb{1}_{\|x+y\|_{\mathbb{H}} \le 2M} \mathbb{1}_{\|x\|_{\mathbb{H}} > M} \mathbb{1}_{\|y\|_{\mathbb{H}} > M} \\ & \le 5M^2 \mathbb{1}_{\|x\|_{\mathbb{H}} > M} + 9M^2 \mathbb{1}_{\|y\|_{\mathbb{H}} > M} \,. \end{aligned} \tag{5.4}$$

In addition setting for any real $u \ge 0$ and any $T > 0$, $g_T(u) = u^2 \wedge T^2$, we have

$$\Big| \|x+y\|_{\mathbb{H}}^2 \mathbb{1}_{\|x+y\|_{\mathbb{H}} \le 2M} - g_{2M}\big(\|x+y\|_{\mathbb{H}}\big) \Big| \le 4M^2 \mathbb{1}_{\|x+y\|_{\mathbb{H}} > 2M} \,.$$

Using these inequalities and the stationarity of $(X_i)_{i \in \mathbb{Z}}$, it follows that

$$\sup_{j \ge k \ge q} \|4 < X'_j, X'_k > - g_{2M}\big(\|X_j + X_k\|_{\mathbb{H}}\big) + g_{2M}\big(\|X_j - X_k\|_{\mathbb{H}}\big)\|_1 \le 44 \times M \mathbb{E}(\|X_0\|_{\mathbb{H}} \mathbb{1}_{\|X_0\|_{\mathbb{H}} > M}) \,.$$

Hence

$$\mathbb{P}\Big(\sum_{i=1}^{[[n/q]/2]} \mathbb{E}(\|\tilde{U}_{2i}\|_{\mathbb{H}}^2 | \mathcal{F}^U_{2(i-1)}) \geq 2s_n^2 \Big) \leq \frac{23nq}{2s_n^2} M \mathbb{E}(\|X_0\|_{\mathbb{H}} \mathbb{1}_{\|X_0\|_{\mathbb{H}} > M})$$
$$+ \frac{nq}{8s_n^2} \sup_{j \geq k \geq q} \|\mathbb{E}\big(g_{2M}(\|X_j + X_k\|_{\mathbb{H}}) | \mathcal{F}_0\big) - \mathbb{E}\big(g_{2M}(\|X_j + X_k\|_{\mathbb{H}})\big)\|_1$$
$$+ \frac{nq}{8s_n^2} \sup_{j \geq k \geq q} \|\mathbb{E}\big(g_{2M}(\|X_j - X_k\|_{\mathbb{H}}) | \mathcal{F}_0\big) - \mathbb{E}\big(g_{2M}(\|X_j - X_k\|_{\mathbb{H}})\big)\|_1 .$$

Since g_{2M} is $2M$-Lipschitz, it follows that

$$\sup_{j \geq k \geq q} \|\mathbb{E}\big(g_{2M}(\|X_j + X_k\|_{\mathbb{H}}) | \mathcal{F}_0\big) - \mathbb{E}\big(g_{2M}(\|X_j + X_k\|_{\mathbb{H}})\big)\|_1 \leq 2M\theta_2(q) ,$$

and the same holds true with $\|X_j + X_k\|_{\mathbb{H}}$ in place of $\|X_j - X_k\|_{\mathbb{H}}$. Consequently

$$\mathbb{P}\Big(\sum_{i=1}^{[[n/q]/2]} \mathbb{E}(\|\tilde{U}_{2i}\|_{\mathbb{H}}^2 | \mathcal{F}^U_{2(i-1)}) \geq 2s_n^2 \Big) \leq \frac{nq}{2s_n^2} M\theta_2(q) + \frac{23nq}{2s_n^2} M \mathbb{E}(\|X_0\|_{\mathbb{H}} \mathbb{1}_{\|X_0\|_{\mathbb{H}} > M}) . \tag{5.5}$$

Now by using Markov's inequality, we get that

$$\mathbb{P}\Big(\sum_{i=3}^{[n/q]} \|U_i' - \tilde{U}_i\|_{\mathbb{H}} + \sum_{k=1}^{n} \|X_k''\|_{\mathbb{H}} \geq x \Big) \leq \frac{1}{x} \Big(\sum_{i=1}^{[n/q]} \mathbb{E}\|\mathbb{E}(U_i' | \mathcal{M}_{(i-2)q})\|_{\mathbb{H}} + \sum_{k=1}^{n} \mathbb{E}\|X_k''\|_{\mathbb{H}} \Big) .$$

Since for every $i \geq 1$, $U_i' = U_i - U_i''$, we get that

$$\mathbb{E}\|\mathbb{E}(U_i' | \mathcal{M}_{(i-2)q})\|_{\mathbb{H}} \leq \mathbb{E}\|\mathbb{E}(U_i | \mathcal{M}_{(i-2)q})\|_{\mathbb{H}} + \mathbb{E}\|U_i''\|_{\mathbb{H}} .$$

Consequently, by stationarity,

$$\mathbb{P}\Big(\sum_{i=3}^{[n/q]} \|U_i' - \tilde{U}_i\|_{\mathbb{H}} + \sum_{k=1}^{n} \|X_k''\|_{\mathbb{H}} \geq x \Big) \leq \frac{n}{x} \Big(\gamma(q) + \mathbb{E}(\|X_0\|_{\mathbb{H}} \mathbb{1}_{\|X_0\|_{\mathbb{H}} > M}) \Big) . \tag{5.6}$$

Starting from (5.2), if q and M are chosen in such a way that $qM \leq x/r \leq x$, we derive from (5.3), (5.5) and (5.6) that

$$\begin{aligned} \mathbb{P}\Big(\max_{1 \leq k \leq n} \|S_k\|_{\mathbb{H}} \geq 4x \Big) \quad &\leq \quad 4 \exp\left(-\frac{r^2 s_n^2}{2x^2} h\left(\frac{x^2}{r s_n^2} \right) \right) + \frac{nx}{r s_n^2} \theta_2(q) + \frac{n}{x} \gamma(q) \\ &\quad + \big(\frac{n}{x} + \frac{23nx}{r s_n^2} \big) \mathbb{E}(\|X_0\|_{\mathbb{H}} \mathbb{1}_{\|X_0\|_{\mathbb{H}} > M}) . \end{aligned} \tag{5.7}$$

Now choose $v = S(x/r)$, $q = \min\{q \in \mathbb{N} : \lambda_2(q) \leq \int_0^v Q(u)du := H(v)\} \wedge n$ and $M = Q(v)$. Since R is right continuous, we get that

$$qM = R(v) = R(S(x/r)) \leq x/r \leq x .$$

Note also that

$$\mathbb{E}(\|X_0\|_{\mathbb{H}}\mathbb{I}_{\|X_0\|_{\mathbb{H}}>M}) \le \int_0^v Q(u)du\,. \tag{5.8}$$

If $q < n$ then the choice of q implies that $\lambda_2(q) \le \int_0^v Q(u)du$. It follows that the inequality is established by also taking into account (5.8) in (5.7). Now if $q = n$ we may have $\lambda_2(q) > v$. However since $\max_{1\le k\le n}|S_k| \le qM + \sum_{k=1}^n |X_k''|$ and $qM \le x$ we have

$$\mathbb{P}\big(\max_{1\le k\le n}\|S_k\|_{\mathbb{H}} \ge 4x\big) \le \frac{n}{x}\mathbb{E}\left(\|X_0\|_{\mathbb{H}}\mathbb{I}_{\|X_0\|_{\mathbb{H}}>M}\right)\,,$$

which implies the desired inequality by using (5.8).

5.2 Proof of Theorem 1

We only sketch the proof since it follows closely that of Theorem 3. We keep the same notations as in the proof of Theorem 3. Since $\|X_0\|_{\mathbb{H}} \le M$, we have $X_i' = X_i$ and $X_i'' = 0$.

We start from the decomposition (5.1), and we still have the upper bound (5.3). Taking $s_n^2 \ge n\sum_{i=0}^{n-1}|\mathbb{E}(< X_0, X_i >)|$, we obtain the upper bound

$$\mathbb{P}\Big(\sum_{i=1}^{[[n/q]/2]}\mathbb{E}(\|\tilde{U}_{2i}\|_{\mathbb{H}}^2|\mathcal{F}_{2(i-1)}^U) \ge 2s_n^2\Big) \le \frac{nq}{2s_n^2}M\delta(q)\,, \tag{5.9}$$

where $\delta(n)$ has been defined in Theorem 1. Since $U_i = U_i'$, instead of (5.6), we have

$$\mathbb{P}\Big(\sum_{i=3}^{[n/q]}\|U_i - \tilde{U}_i\|_{\mathbb{H}} \ge x\Big) \le \frac{n}{x}\delta(q)\,. \tag{5.10}$$

From (5.3), (5.9) and (5.10), we obtain

$$\mathbb{P}\Big(\max_{1\le k\le n}\|S_k\|_{\mathbb{H}} \ge 4x\Big) \le 4\exp\left(-\frac{s_n^2}{2(qM)^2}h\left(\frac{xqM}{s_n^2}\right)\right) + \Big\{\frac{nqM}{s_n^2} + \frac{n}{x}\Big\}\delta(q)\,. \tag{5.11}$$

Taking $q = [x/rM]$, the result follows.

5.3 Proof of Theorem 4

Let us first prove the inequality (3.15) with $A = 8\sqrt{2}\big(\sum_{i=0}^{\infty}\int_0^{\gamma(i)} Q\circ G(u)du\big)^{1/2}$. We follow the proof of Theorem 6.4 page 89 in Rio (2000), and we use the same notations: $Lx = \ln(x\vee e)$ and $LLx = \ln(\ln(x\vee e)\vee e)$. We apply Inequality (3.13) with

$$r = r_n = 4LLn, \quad x = x_n = (A\sqrt{nLLn})/4 \quad \text{and} \quad s_n = x_n/\sqrt{2r_n}\,.$$

We obtain

$$\sum_{n>0}\frac{1}{n}\mathbb{P}\Big(\sup_{1\le k\le n}|S_k|\ge A\sqrt{nLLn}\Big)\le 4\sum_{n>0}\frac{1}{n3^{LLn}}+50\sum_{n>0}\frac{1}{x_n}\int_0^{S(x_n/r_n)}Q(u)du\,.$$

Clearly the first series on right hand converges. From the end of the proof of Theorem 6.4 in Rio (2000), we see that the second series on the right hand side converges as soon as (3.14) holds. This completes the proof of (3.15).

To prove the almost sure invariance principle, we first prove that

$$\text{there exists a mean zero gaussian measure } \nu \text{ with covariance function } \Gamma\,, \tag{5.12}$$

and that

$$\text{the sequence } \{S_n/\sqrt{nLLn}\,,\, n\ge 1\} \text{ is almost surely relatively compact in } \mathbb{H}. \tag{5.13}$$

According to Corollary 2(β) in Dedecker and Merlevède (2003), the condition

$$\sum_{k>0}\int_0^{\gamma(k)}Q\circ G(u)du<\infty\,, \tag{5.14}$$

(which is clearly weaker than (3.14)) implies that the sequence $n^{-1/2}S_n$ converges in distribution to $\mathcal{N}(0,\Gamma)$ where the operator $\Gamma\in\mathcal{S}(\mathbb{H})$ is defined by (2.8). This proves (5.12).

We turn now to the proof of (5.13). With this aim, we argue as page 698 in Dehling and Philipp (1982, proof of their Theorem 1), with the help of (3.15). Let $\{e_i, i\ge 1\}$ be a complete orthonormal basis for $\mathbb{H}$. We write for each $k\in\mathbb{Z}$

$$X_k=\sum_{i\ge 1}<X_k,e_i>e_i \quad\text{and}\quad P_N(X_k)=\sum_{i=1}^{N}<X_k,e_i>e_i\,.$$

Applying (3.15) to the sequence $\{X_k-P_N(X_k), k\in\mathbb{Z}\}$, we get that with probability one

$$\limsup_{n\to\infty}\frac{\|\sum_{k=1}^n(X_k-P_N(X_k))\|_{\mathbb{H}}}{\sqrt{nLLn}}\le A_N\,, \tag{5.15}$$

where

$$A_N=8\sqrt{2}\Big(\sum_{i=0}^{\infty}\int_0^{\gamma_N(i)}Q_N\circ G_N(u)du\Big)^{1/2},$$

with $Q_N=Q_{\|(I-P_N)(X_0)\|_{\mathbb{H}}}$, $H_N(x)=\int_0^x Q_N(u)du$, $G_N=H_N^{-1}$, $\gamma_N(i)=\mathbb{E}\|\mathbb{E}((I-P_N)(X_i)|\mathcal{F}_0\|_{\mathbb{H}}$. Note that $\gamma_N(i)\le\gamma(i)$, $Q_N\le Q$, and $G_N\ge G$. Hence, since Q_N is non increasing,

$$A_N\le 8\sqrt{2}\Big(\sum_{i=0}^{\infty}\int_0^{\gamma(i)}Q_N\circ G(u)du\Big)^{1/2}.$$

Since for any $u \in [0,1]$, $Q_N(u) \leq Q(u)$ and $\lim_{N\to\infty} Q_N = 0$, we get by using (3.14) and the Lebesgue dominated convergence theorem that for each $\rho > 0$ there is an integer $N_0(\rho)$ such that for all $N \geq N_0(\rho)$, we have $A_N \leq \rho$. Hence with probability one,

$$\limsup_{n\to\infty} \frac{\| \sum_{k=1}^n (X_k - P_N(X_k)) \|_{\mathbb{H}}}{\sqrt{nLLn}} \leq \rho\,. \tag{5.16}$$

Now applying again (3.15), we get that the sequence

$$\Big\{ \frac{\sum_{k=1}^n P_{N_0(\rho)}(X_k)}{\sqrt{nLLn}}, n \geq 1 \Big\}$$

is with probability one relatively compact. This fact combining with (5.16) establishes (5.13).

To finish the proof of Theorem 4, we shall use a variant of Theorem 3.2 in Berger (1990). In this theorem, it is proved that if $(X_k, \mathcal{F}_k)$ has the *weak M_2 property* then (5.12) and (5.13) together are equivalent to the almost sure invariance principle (3.16). In the context of Hilbert spaces the *weak M_2 property* means exactly that: for any $x \in \mathbb{H}$

$$< X_0, x >= d_0(x) + Z_0(x) - Z_0(x) \circ T\,, \tag{5.17}$$

where $\mathbb{E}(Z_0^2(x)) < \infty$, $\mathbb{E}(d_0^2(x)) < \infty$, $d_0(x)$ is $\mathcal{F}_0$-measurable, and $\mathbb{E}(d_0(x)|\mathcal{F}_{-1}) = 0$. Since $Z_0(x)$ is in $\mathbb{L}^2$, then

$$\frac{Z_0(x) \circ T^n}{\sqrt{n}} \quad \text{converges to 0 almost surely,}$$

and the almost sure limit behavior of $< S_n, x > / \sqrt{n \ln(\ln(n))}$ can be deduced from the almost sure limit behavior of $M_n(x)/\sqrt{n \ln(\ln(n))}$, where $M_n(x)$ is the martingale defined by $M_n(x) = \sum_{i=1}^n d_0(x) \circ T^i$.

In fact, Berger's proof works perfectly if instead of the *weak M_2 property* we have the following decomposition: for any $x \in \mathbb{H}$

$$\text{(5.17) holds with } \mathbb{E}(d_0^2(x)) < \infty,\ d_0(x) \text{ is } \mathcal{F}_0\text{-measurable and } \mathbb{E}(d_0(x)|\mathcal{F}_{-1}) = 0, \tag{5.18}$$

and $Z_0(x) \circ T^n$ is such that

$$\frac{Z_0(x) \circ T^n}{\sqrt{nLLn}} \quad \text{converges to 0 almost surely.} \tag{5.19}$$

Hence, the proof of the almost sure invariance principle will be complete if we can prove (5.18) and (5.19).

We first prove (5.18). In 1973, Gordin (see also Esseen and Janson (1985)) proved that, if T is ergodic and

$$\sum_{k\geq 1} \|\mathbb{E}(< X_k, x > |\mathcal{F}_0)\|_1 < \infty\,, \tag{5.20}$$

and

$$\limsup_{n\to\infty} \frac{1}{\sqrt{n}} \mathbb{E}\Big(\Big| \sum_{k=1}^{n} < X_k, x > \Big| \Big) < \infty \,, \tag{5.21}$$

then (5.18) holds with $\mathbb{E}(d_0^2(x)) < \infty$ and $\mathbb{E}(|Z_0(x)|) < \infty$. Now

$$\|\mathbb{E}(< X_k, x > |\mathcal{F}_0)\|_1 \le \mathbb{E}(\|\mathbb{E}(X_k|\mathcal{F}_0)\|_{\mathbb{H}}) = \gamma(k) \,.$$

Hence (5.20) holds as soon as $\sum_{k>0} \gamma(k) < \infty$, which is true under (3.14). Now clearly (5.21) holds as soon as $\sum_{k=0}^{\infty} |\mathrm{Cov}(< X_0, x >, < X_k, x >)| < \infty$. But this again follows from (3.14) by applying inequality (3.33) in Dedecker and Merlevède (2003). This completes the proof of (5.18).

It remains to prove (5.19). According to the lemma page 428 in Volný and Samek (2000), we have either (5.19) or

$$\mathbb{P}\Big(\limsup_{n\to\infty} \frac{|Z_0(x) \circ T^n|}{\sqrt{nLLn}} = \infty \Big) = 1 \,. \tag{5.22}$$

From (5.18), we have that

$$< S_n, x >= M_n(x) + Z_0(x) \circ T - Z_0(x) \circ T^{n+1} \,. \tag{5.23}$$

Using the decomposition (5.23), the fact that $M_n(x)$ satisfies the law of the iterated logarithm and that S_n satisfies (3.15), it is clear that (5.22) cannot hold, which then proves (5.19). The proof of (3.16) is complete.

5.4 Proof of Theorem 2

We only sketch the proof since it follows closely that of Theorem 4. We keep the same notations as in the proof of Theorem 4.

To prove item 1, we take the same values for x_n, r_n and s_n as in the proof of Theorem 4. We obtain that

$$\sum_{n>0} \frac{1}{n} \mathbb{P}\Big(\sup_{1\le k\le n} |S_k| \ge A\sqrt{nLLn} \Big) \le 4 \sum_{n>0} \frac{1}{n3^{LLn}} + 12 \sum_{n>0} \frac{1}{A\sqrt{nLLn}} \delta([A\sqrt{n}/16M\sqrt{LLn}]) \,,$$

and the second series on right hand is finite as soon as $\sum_{k>0} \delta(k) < \infty$, which is equivalent to (2.6) and (2.7).

Item 2 follows by noting that, in the bounded case, (5.14) holds as soon as (2.6) holds.

To prove Item 3, we first prove (5.16). Note first that (2.7) and (2.9) together imply that that, for any positive integer N,

$$\sum_{n>0} \sup_{i\ge j\ge n} \|\mathbb{E}(< X_i - P_N(X_i), X_j - P_N(X_j) > |\mathcal{F}_0) - \mathbb{E}(< X_i - P_N(X_i), X_j - P_N(X_j) >)\|_1 < \infty \,. \tag{5.24}$$

This means that item 1 hold for the sequence $(X_i - P_N(X_i))_{i\in\mathbb{Z}}$, so that (5.15) holds with

$$\begin{aligned} A_N &= 8\sqrt{2}\Big(\mathbb{E}(\|X_0 - P_N(X_0)\|^2_{\mathbb{H}}) + \sum_{k>0} |\mathbb{E}(< X_0 - P_N(X_0), X_k - P_N(X_k) >)|\Big)^{1/2} \\ &\leq 8\sqrt{2}\Big(M \sum_{k\geq 0} \mathbb{E}(\|\mathbb{E}(X_k - P_N(X_k)|\mathcal{F}_0)\|_{\mathbb{H}})\Big)^{1/2} . \end{aligned}$$

Since $\mathbb{E}(\|\mathbb{E}(X_k - P_N(X_k)|\mathcal{F}_0)\|_{\mathbb{H}}) \leq \mathbb{E}(\|\mathbb{E}(X_k|\mathcal{F}_0)\|_{\mathbb{H}})$ and since (2.6) holds, it follows from the dominated convergence theorem that A_N tends to zero as N tends to infinity, so that (5.16) is satisfied.

Starting from (5.16), the end of the proof of Theorem 2 is the same as that of Theorem 4.

5.5 Proof of Corollaries 2 and 3

Let $X_k = f(Y_k) - \mathbb{E}(f(Y_k))$. In both cases, we have to check (3.14). Applying the coupling result given in Dedecker and Merlevède (2006, Lemma 1) (see also Proposition 4 in Rüschendorf (1985)), we infer that there exists $\bar{Y}_n$ distributed as Y_n and independent of $\mathcal{F}_0$ such that

$$\mathbb{E}(|Y_n - \bar{Y}_n|^\alpha_{\mathbb{B}}) = \tau_{1,\alpha}(n) \leq \tau_{2,\alpha}(n) .$$

In the same way, for $n \leq i < j$ there exists (Y_i^*, Y_j^*) distributed as (Y_i, Y_j) and independent of $\mathcal{F}_0$ such that

$$\frac{1}{2}\mathbb{E}(|Y_i - Y_i^*|^\alpha_{\mathbb{B}} + |Y_j - Y_j^*|^\alpha_{\mathbb{B}}) = \Big\| \sup_{f\in\Lambda_1(\mathbb{B}^2,\alpha)} \Big|\mathbb{E}(f(Y_i, Y_j)|\mathcal{F}_0) - \mathbb{E}(f(Y_i, Y_j))\Big|\Big\|_1 \leq \tau_{2,\alpha}(i) \leq \tau_{2,\alpha}(n) .$$

Clearly

$$\gamma(n) = \mathbb{E}(\|\mathbb{E}(f(Y_n)|\mathcal{F}_0) - \mathbb{E}(f(Y_n))\|_{\mathbb{H}}) \leq \mathbb{E}(\|f(Y_n) - f(\bar{Y}_n)\|_{\mathbb{H}}) .$$

Consequently, if $f \in H_\alpha$, one has $\gamma(n) \leq K\mathbb{E}(|Y_n - \bar{Y}_n|^\alpha_{\mathbb{B}}) = K\tau_{1,\alpha}(n)$. Now, if $\alpha = 1$ and $f \in \mathcal{L}_c$, one has

$$\gamma(n) \leq K\mathbb{E}(c(|Y_n - \bar{Y}_n|_{\mathbb{B}})) \leq Kc(\mathbb{E}(|Y_n - \bar{Y}_n|_{\mathbb{B}})) = Kc(\tau_{1,1}(n)) .$$

In the same way, if g is in $\Lambda_1(\mathbb{H})$,

$$\|\mathbb{E}(g(X_i + X_j)|\mathcal{F}_0) - \mathbb{E}(g(X_i + X_j))\|_1 \leq \mathbb{E}(\|f(Y_i) - f(Y_i^*)\|_{\mathbb{H}} + \|f(Y_j) - f(Y_j^*)\|_{\mathbb{H}}) .$$

Hence, if $f \in H_\alpha$ and $n \leq i < j$,

$$\|\mathbb{E}(g(X_i + X_j)|\mathcal{F}_0) - \mathbb{E}(g(X_i + X_j))\|_1 \leq 2\tau_{2,\alpha}(n) ,$$

and if $\alpha = 1$ and $f \in \mathcal{L}_c$,

$$\|\mathbb{E}(g(X_i + X_j)|\mathcal{F}_0) - \mathbb{E}(g(X_i + X_j))\|_1 \leq 2Kc(\tau_{2,1}(n)) .$$

Note that the same inequalities hold with $X_i - X_j$ instead of $X_i + X_j$.

As a consequence, we obtain that:

1. If $f \in H_\alpha$, then $\lambda_2(n) \leq 2K\tau_{2,\alpha}(n)$.
2. If $f \in \mathcal{L}_c$, then $\lambda_2(n) \leq 2Kc(\tau_{2,1}(n))$.

Corollary 2 follows from item 1, and Corollary 3 follows from item 2.

6 Appendix

In this section, we recall the following consequence of Theorem 3.4 given in Pinelis (1994).

Lemma 1. *Let $(\mathbb{H}, \|\cdot\|_{\mathbb{H}})$ be a real separable Hilbert space. Let $\{d_j, \mathcal{F}_j\}_{j\geq 1}$ be a sequence of $\mathbb{H}$-valued martingale differences with $\|d_j\|_{\mathbb{H}} \leq c$. Set $M_j = \sum_{i=1}^{j} d_i$. Then for all $x, y > 0$,*

$$\mathbb{P}\left(\sup_{1\leq j\leq n} \|M_j\|_{\mathbb{H}} \geq x, \sum_{j=1}^{n} \mathbb{E}(\|d_j\|_{\mathbb{H}}^2 | \mathcal{F}_{j-1}) \leq y\right) \leq 2\exp\left(-\frac{y}{c^2}\, h\Big(\frac{xc}{y}\Big)\right),$$

where $h(u) := (1+u)\ln(1+u) - u$.

References

[1] Berger, E. (1990). An almost sure invariance principle for stationary ergodic sequences of Banach space valued random variables, *Probab. Theory Relat. Fields* **84** 161-201.

[2] Dedecker, J. and Merlevède, F. (2003). The conditional central limit theorem in Hilbert spaces, *Stochastic Process Appl.* **108** 229-262.

[3] Dedecker, J. and Merlevède, F. (2006). Inequalities for partial sums of Hilbert valued dependent sequences and applications, *Mathematical methods of Statistics* **15** 176-206.

[4] Dedecker, J. and Prieur, C. (2007). An empirical central limit theorem for dependent sequences, *Stochastic Process. Appl.* **117** 121-142.

[5] Dehling, H. (1983). A note on a theorem of Berkes and Philipp, *Z. Wahrsch. Verw. Gebiete* **62** 39-42.

[6] Dehling, H. and Philipp, W. (1982). Almost sure invariance principles for weakly dependent vector-valued random variables. *Ann. Probab.* **10** 689-701.

[7] Esseen, C.-G. and Janson, S. (1985). On moment conditions for normed sums of independent variables and martingale differences. *Stochastic Process. Appl.* **19** 173-182.

[8] Gordin, M. I. (1973). Abstracts of communication, T.1:A-K, International conference on probability theory, Vilnius.

[9] Kuksin, S., Piatnitski, A. and Shirikyan, A. (2002). A coupling approach to randomly forced nonlinear PDE's. II. *Commun. Math. Phys.* **230** 81-85.

[10] Lindenstrauss, J. and Tzafriri, L. (1977) Classical Banach spaces. I. Sequence spaces, *Ergebnisse der Mathematik und ihrer Grenzgebiete* **92**. Springer-Verlag, Berlin-New York.

[11] Masmoudi, N. and Young L.-S. (2002). Ergodic theory of infinite dimensional systems with applications to dissipative parabolic PDE's. *Commun. Math. Phys.* **227** 461-481.

[12] Merlevède, F. (2008). On a maximal inequality for strongly mixing random variables in Hilbert spaces. Application to the compact law of the iterated logarithm. *Publ. Inst. Stat. Univ. Paris* **52** 47-60.

[13] Pinelis, I. (1994). Optimum bounds for the distributions of martingales in Banach spaces. *Ann. Probab.*, **22** 1679-1706.

[14] Rio, E. (1995). The functional law of the iterated logarithm for stationary strongly mixing sequences. *Ann. Probab.* **23** 1188–1203.

[15] Rosenblatt, M. (1956). A central limit theorem and a strong mixing condition, *Proc. Nat. Acad. Sci. U. S. A.* **42** 43-47.

[16] Rüschendorf, L. (1985). The Wasserstein Distance and Approximation Theorems. *Z. Wahrsch. Verw. Gebiete* **70** 117-129.

[17] Shirikyan, A. (2006). Law of large numbers and central limit theorem for randomly forced PDE's. *Probab. Theory Relat. Fields* **134** 215-247.

[18] Volný, D. and Samek, P. (2000). On the invariance principle and the law of iterated logarithm for stationary processes. *Mathematical physics and stochastic analysis* 424-438, World Sci. Publishing, River Edge.