Rates of convergence for minimal distances in the central limit theorem under projective criteria

Jérôme Dedecker [a], Florence Merlevède [b] *and* Emmanuel Rio [c]

[a] Université Paris 6, LSTA, 175 rue du Chevaleret, 75013 Paris, FRANCE.
E-mail: dedecker@ccr.jussieu.fr

[b] Université Paris 6, LPMA and C.N.R.S UMR 7599, 175 rue du Chevaleret, 75013 Paris, FRANCE. E-mail: merleve@ccr.jussieu.fr

[c] Université de Versailles, Laboratoire de mathématiques, UMR 8100 CNRS, Bâtiment Fermat, 45 Avenue des Etats-Unis, 78035 Versailles, FRANCE. E-mail: rio@math.uvsq.fr

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Abstract

In this paper, we give estimates of ideal or minimal distances between the distribution of the normalized partial sum and the limiting Gaussian distribution for stationary martingale difference sequences or stationary sequences satisfying projective criteria. Applications to functions of linear processes and to functions of expanding maps of the interval are given.

1 Introduction and Notations

Let $X_1, X_2, \dots$ be a strictly stationary sequence of real-valued random variables (r.v.) with mean zero and finite variance. Set $S_n = X_1 + X_2 + \cdots + X_n$. By $P_{n^{-1/2}S_n}$ we denote the law of $n^{-1/2}S_n$ and by G_{σ^2} the normal distribution $N(0, \sigma^2)$. In this paper, we shall give quantitative estimates of the approximation of $P_{n^{-1/2}S_n}$ by G_{σ^2} in terms of minimal or ideal metrics.

Let $\mathcal{L}(\mu, \nu)$ be the set of the probability laws on $\mathbb{R}^2$ with marginals μ and ν. Let us consider the following minimal distances (sometimes called Wasserstein distances of order r)

$$W_r(\mu,\nu) = \begin{cases} \inf\Big\{ \int |x-y|^r P(dx,dy) : P \in \mathcal{L}(\mu,\nu) \Big\} & \text{if } 0 < r < 1 \\ \inf\Big\{ \Big(\int |x-y|^r P(dx,dy) \Big)^{1/r} : P \in \mathcal{L}(\mu,\nu) \Big\} & \text{if } r \geq 1\,. \end{cases}$$

It is well known that for two probability measures μ and ν on $\mathbb{R}$ with respective distributions functions (d.f.) F and G,

$$W_r(\mu,\nu) = \Big(\int_0^1 |F^{-1}(u) - G^{-1}(u)|^r du \Big)^{1/r} \text{ for any } r \geq 1. \tag{1.1}$$

We consider also the following ideal distances of order r (Zolotarev distances of order r). For two probability measures μ and ν, and r a positive real, let

$$\zeta_r(\mu,\nu) = \sup \Big\{ \int f d\mu - \int f d\nu : f \in \Lambda_r \Big\},$$

where Λ_r is defined as follows: denoting by l the natural integer such that $l < r \leq l+1$, Λ_r is the class of real functions f which are l-times continuously differentiable and such that

$$|f^{(l)}(x) - f^{(l)}(y)| \leq |x-y|^{r-l} \text{ for any } (x,y) \in \mathbb{R} \times \mathbb{R}\,. \tag{1.2}$$

It follows from the Kantorovich-Rubinstein theorem (1958) that for any $0 < r \leq 1$,

$$W_r(\mu,\nu) = \zeta_r(\mu,\nu)\,. \tag{1.3}$$

For probability laws on the real line, Rio (1998) proved that for any $r > 1$,

$$W_r(\mu,\nu) \leq c_r \big(\zeta_r(\mu,\nu)\big)^{1/r}, \tag{1.4}$$

where c_r is a constant depending only on r.

For independent random variables, Ibragimov (1966) established that if $X_1 \in \mathbb{L}^p$ for $p \in]2,3]$, then $W_1(P_{n^{-1/2}S_n}, G_{\sigma^2}) = O(n^{1-p/2})$ (see his Theorem 4.3). Still in the case of independent r.v.'s, Zolotarev (1976) obtained the following upper bound for the ideal distance: if $X_1 \in \mathbb{L}^p$ for $p \in]2,3]$, then $\zeta_p(P_{n^{-1/2}S_n}, G_{\sigma^2}) = O(n^{1-p/2})$. From (1.4), the result of Zolotarev entails that, for $p \in]2,3]$, $W_p(P_{n^{-1/2}S_n}, G_{\sigma^2}) = O(n^{1/p-1/2})$ (which was obtained by Sakhanenko (1985) for any $p > 2$). From (1.1) and Hölder's inequality, we easily get that for independent random variables in $\mathbb{L}^p$ with $p \in]2,3]$,

$$W_r(P_{n^{-1/2}S_n}, G_{\sigma^2}) = O(n^{-(p-2)/2r}) \quad \text{for any } 1 \leq r \leq p. \tag{1.5}$$

In this paper, we are interested in extensions of (1.5) to sequences of dependent random variables. More precisely, for $X_1 \in \mathbb{L}^p$ and p in $]2,3]$ we shall give $\mathbb{L}^p$-projective criteria under which: for $r \in [p-2, p]$ and $(r,p) \neq (1,3)$,

$$W_r(P_{n^{-1/2}S_n}, G_{\sigma^2}) = O(n^{-(p-2)/2\max(1,r)})\,. \tag{1.6}$$

As we shall see in Remark 2.3, (1.6) applied to $r = p - 2$ provides the rate of convergence $O(n^{-\frac{p-2}{2(p-1)}})$ in the Berry-Esseen theorem.

When $(r, p) = (1, 3)$, Dedecker and Rio (2007) obtained that $W_1(P_{n^{-1/2}S_n}, G_{\sigma^2}) = O(n^{-1/2})$ for stationary sequences of random variables in $\mathbb{L}^3$ satisfying $\mathbb{L}^1$ projective criteria or weak dependence assumptions (a similar result was obtained by Pène (2005) in the case where the variables are bounded). In this particular case our approach provides a new criterion under which $W_1(P_{n^{-1/2}S_n}, G_{\sigma^2}) = O(n^{-1/2} \log n)$.

Our paper is organized as follows. In Section 2, we give projective conditions for stationary martingales differences sequences to satisfy (1.6) in the case $(r, p) \neq (1, 3)$. To be more precise, let $(X_i)_{i\in\mathbb{Z}}$ be a stationary sequence of martingale differences with respect to some σ-algebras $(\mathcal{F}_i)_{i\in\mathbb{Z}}$ (see Section 1.1 below for the definition of $(\mathcal{F}_i)_{i\in\mathbb{Z}}$). As a consequence of our Theorem 2.1, we obtain that if $(X_i)_{i\in\mathbb{Z}}$ is in $\mathbb{L}^p$ with $p \in]2, 3]$ and satisfies

$$\sum_{n=1}^{\infty} \frac{1}{n^{2-p/2}} \Big\| \mathbb{E}\Big(\frac{S_n^2}{n}\Big|\mathcal{F}_0\Big) - \sigma^2 \Big\|_{p/2} < \infty\,, \tag{1.7}$$

then the upper bound (1.6) holds provided that $(r, p) \neq (1, 3)$. In the case $r = 1$ and $p = 3$, we obtain the upper bound $W_1(P_{n^{-1/2}S_n}, G_{\sigma^2}) = O(n^{-1/2} \log n)$.

In Section 3, starting from the coboundary decomposition going back to Gordin (1969), and using the results of Section 2, we obtain $\mathbb{L}^p$-projective criteria ensuring (1.6) (if $(r, p) \neq (1, 3)$). For instance, if $(X_i)_{i\in\mathbb{Z}}$ is a stationary sequence of $\mathbb{L}^p$ random variables adapted to $(\mathcal{F}_i)_{i\in\mathbb{Z}}$, we obtain (1.6) for any $p \in]2, 3[$ and any $r \in [p - 2, p]$ provided that (1.7) holds and the series $\mathbb{E}(S_n|\mathcal{F}_0)$ converge in $\mathbb{L}^p$. In the case where $p = 3$, this last condition has to be strengthened. Our approach makes also possible to treat the case of non-adapted sequences.

Section 4 is devoted to applications. In particular, we give sufficient conditions for some functions of Harris recurrent Markov chains and for functions of linear processes to satisfy the bound (1.6) in the case $(r, p) \neq (1, 3)$ and the rate $O(n^{-1/2} \log n)$ when $r = 1$ and $p = 3$. Since projective criteria are verified under weak dependence assumptions, we give an application to functions of ϕ-dependent sequences in the sense of Dedecker and Prieur (2007). These conditions apply to unbounded functions of uniformly expanding maps.

1.1 Preliminary notations

Throughout the paper, Y is a $N(0, 1)$-distributed random variable. We shall also use the following notations. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, and $T : \Omega \mapsto \Omega$ be a bijective bimeasurable transformation preserving the probability $\mathbb{P}$. For a σ-algebra $\mathcal{F}_0$ satisfying $\mathcal{F}_0 \subseteq T^{-1}(\mathcal{F}_0)$, we define the nondecreasing filtration $(\mathcal{F}_i)_{i\in\mathbb{Z}}$ by $\mathcal{F}_i = T^{-i}(\mathcal{F}_0)$. Let $\mathcal{F}_{-\infty} = \bigcap_{k\in\mathbb{Z}} \mathcal{F}_k$ and

$\mathcal{F}_\infty = \bigvee_{k\in\mathbb{Z}} \mathcal{F}_k$. We shall denote sometimes by $\mathbb{E}_i$ the conditional expectation with respect to $\mathcal{F}_i$. Let X_0 be a zero mean random variable with finite variance, and define the stationary sequence $(X_i)_{i\in\mathbb{Z}}$ by $X_i = X_0 \circ T^i$.

2 Stationary sequences of martingale differences.

In this section we give bounds for the ideal distance of order r in the central limit theorem for stationary martingale differences sequences $(X_i)_{i\in\mathbb{Z}}$ under projective conditions.

Notation 2.1. For any $p > 2$, define the envelope norm $\|\,.\,\|_{1,\Phi,p}$ by

$$\|X\|_{1,\Phi,p} = \int_0^1 (1 \vee \Phi^{-1}(1-u/2))^{p-2} Q_X(u)du$$

where Q_X denotes the quantile function of $|X|$, and Φ denotes the d.f. of the $N(0,1)$ law.

Theorem 2.1. *Let $(X_i)_{i\in\mathbb{Z}}$ be a stationary martingale differences sequence with respect to $(\mathcal{F}_i)_{i\in\mathbb{Z}}$. Let σ denote the standard deviation of X_0. Let $p \in]2,3]$. Assume that $\mathbb{E}|X_0|^p < \infty$ and that*

$$\sum_{n=1}^{\infty} \frac{1}{n^{2-p/2}} \Big\| \mathbb{E}\Big(\frac{S_n^2}{n}\Big|\mathcal{F}_0\Big) - \sigma^2 \Big\|_{1,\Phi,p} < \infty\,, \tag{2.1}$$

and

$$\sum_{n=1}^{\infty} \frac{1}{n^{2/p}} \Big\| \mathbb{E}\Big(\frac{S_n^2}{n}\Big|\mathcal{F}_0\Big) - \sigma^2 \Big\|_{p/2} < \infty\,. \tag{2.2}$$

Then, for any $r \in [p-2,p]$ with $(r,p) \neq (1,3)$, $\zeta_r(P_{n^{-1/2}S_n}, G_{\sigma^2}) = O(n^{1-p/2})$, and for $p=3$ $\zeta_1(P_{n^{-1/2}S_n}, G_{\sigma^2}) = O(n^{-1/2}\log n)$.

Remark 2.1. Under the assumptions of Theorem 2.1, $\zeta_r(P_{n^{-1/2}S_n}, G_{\sigma^2}) = O(n^{-r/2})$ if $r < p-2$. Indeed, let $p' = r+2$. Since $p' < p$, if the conditions (2.1) and (2.2) are satisfied for p, they also hold for p'. Hence Theorem 2.1 applies with p'.

From (1.3) and (1.4), the following result holds for the Wasserstein distances of order r.

Corollary 2.1. *Under the conditions of Theorem 2.1, $W_r(P_{n^{-1/2}S_n}, G_{\sigma^2}) = O(n^{-(p-2)/2\max(1,r)})$ for any r in $[p-2,p]$, provided that $(r,p) \neq (1,3)$.*

Remark 2.2. For p in $]2,3]$, $W_p(P_{n^{-1/2}S_n}, G_{\sigma^2}) = O(n^{-(2-p)/2p})$. This bound was obtained by Sakhanenko (1985) in the independent case. For $p < 3$, we have $W_1(P_{n^{-1/2}S_n}, G_{\sigma^2}) = O(n^{1-p/2})$. This bound was obtained by Ibragimov (1966) in the independent case.

Remark 2.3. Let Π_n be the Prokhorov distance between the law of $n^{-1/2}S_n$ and the normal distribution $N(0,\sigma^2)$. From Markov's inequality,

$$\Pi_n \le (W_r(P_{n^{-1/2}S_n}, G_{\sigma^2}))^{1/(r+1)} \text{ for any } 0 < r \le 1\,.$$

Taking $r = p-2$, it follows that under the assumptions of Theorem 2.1,

$$\Pi_n = O(n^{-\frac{p-2}{2(p-1)}}) \text{ if } p < 3 \text{ and } \Pi_n = O(n^{-1/4}\sqrt{\log n}) \quad \text{if } p = 3. \tag{2.3}$$

For p in $]2,4]$, under (2.2), we have that $\|\sum_{i=1}^n \mathbb{E}(X_i^2 - \sigma^2|\mathcal{F}_{i-1})\|_{p/2} = O(n^{2/p})$ (apply Theorem 2 in Wu and Zhao (2006)). Applying then the result in Heyde and Brown (1970), we get that if $(X_i)_{i\in\mathbb{Z}}$ is a stationary martingale difference sequence in $\mathbb{L}^p$ such that (2.2) is satisfied then

$$\|F_n - \Phi_\sigma\|_\infty = O\big(n^{-\frac{p-2}{2(p+1)}}\big)\,.$$

where F_n is the distribution function of $n^{-1/2}S_n$ and Φ_σ is the d.f. of G_{σ^2}. Now

$$\|F_n - \Phi_\sigma\|_\infty \le \big(1 + \sigma^{-1}(2\pi)^{-1/2}\big)\Pi_n\,.$$

Consequently the bounds obtained in (2.3) improve the one given in Heyde and Brown (1970), provided that (2.1) holds.

Remark 2.4. Notice that if $(X_i)_{i\in\mathbb{Z}}$ is a stationary martingale difference sequence in $\mathbb{L}^3$ such that $\mathbb{E}(X_0^2) = \sigma^2$ and

$$\sum_{k>0} k^{-1/2}\|\mathbb{E}(X_k^2|\mathcal{F}_0) - \sigma^2\|_{3/2} < \infty, \tag{2.4}$$

then the conditions (2.1) and (2.2) hold for $p = 3$. Consequently, if (2.4) holds, then Remark 2.3 gives $\|F_n - \Phi_\sigma\|_\infty = O\big(n^{-1/4}\sqrt{\log n}\big)$. This result has to be compared with Theorem 6 in Jan (2001), which states that $\|F_n - \Phi_\sigma\|_\infty = O(n^{-1/4})$ if $\sum_{k>0}\|\mathbb{E}(X_k^2|\mathcal{F}_0) - \sigma^2\|_{3/2} < \infty$.

Remark 2.5. Notice that if $(X_i)_{i\in\mathbb{Z}}$ is a stationary martingale differences sequence, then the conditions (2.1) and (2.2) are respectively equivalent to

$$\sum_{j\ge 0} 2^{j(p/2-1)}\|2^{-j}\,\mathbb{E}(S_{2^j}^2|\mathcal{F}_0) - \sigma^2\|_{1,\Phi,p} < \infty, \text{ and } \sum_{j\ge 0} 2^{j(1-2/p)}\|2^{-j}\,\mathbb{E}(S_{2^j}^2|\mathcal{F}_0) - \sigma^2\|_{p/2} < \infty\,.$$

To see this, let $A_n = \|\mathbb{E}(S_n^2|\mathcal{F}_0\big) - \mathbb{E}(S_n^2)\|_{1,\Phi,p}$ and $B_n = \|\mathbb{E}(S_n^2|\mathcal{F}_0\big) - \mathbb{E}(S_n^2)\|_{p/2}$. We first show that A_n and B_n are subadditive sequences. Indeed, by the martingale property and the stationarity of the sequence, for all positive i and j

$$\begin{aligned} A_{i+j} &= \|\mathbb{E}(S_i^2 + (S_{i+j} - S_i)^2|\mathcal{F}_0\big) - \mathbb{E}(S_i^2 + (S_{i+j} - S_i)^2)\|_{1,\Phi,p} \\ &\le A_i + \|\mathbb{E}\big((S_{i+j} - S_i)^2 - \mathbb{E}(S_j^2)\,|\mathcal{F}_0\big)\|_{1,\Phi,p}\,. \end{aligned}$$

Proceeding as in the proof of (4.6), p. 65 in Rio (2000), one can prove that, for any σ-field $\mathcal{A}$ and any integrable random variable X, $\|\mathbb{E}(X|\mathcal{A})\|_{1,\Phi,p} \le \|X\|_{1,\Phi,p}$. Hence

$$\|\mathbb{E}\big((S_{i+j}-S_i)^2 - \mathbb{E}(S_j^2)\,|\mathcal{F}_0\big)\|_{1,\Phi,p} \le \|\mathbb{E}((S_{i+j}-S_i)^2 - \mathbb{E}(S_j^2)\,|\mathcal{F}_i\big)\|_{1,\Phi,p}\,.$$

By stationarity, it follows that $A_{i+j} \le A_i + A_j$. Similarly $B_{i+j} \le B_i + B_j$. The proof of the equivalences then follows by using the same arguments as in the proof of Lemma 2.7 in Peligrad and Utev (2005).

3 Rates of convergence for stationary sequences

In this section, we give estimates for the ideal distances of order r for stationary sequences which are not necessarily adapted to $\mathcal{F}_i$.

Theorem 3.1. *Let $(X_i)_{i\in\mathbb{Z}}$ be a stationary sequence of centered random variables in $\mathbb{L}^p$ with $p \in]2,3[$, and let $\sigma_n^2 = n^{-1}\mathbb{E}(S_n^2)$. Assume that*

$$\sum_{n>0} \mathbb{E}(X_n|\mathcal{F}_0) \text{ and } \sum_{n>0}(X_{-n} - \mathbb{E}(X_{-n}|\mathcal{F}_0)) \text{ converge in } \mathbb{L}^p\,, \tag{3.1}$$

and

$$\sum_{n\ge 1} n^{-2+p/2} \| \, n^{-1}\,\mathbb{E}(S_n^2|\mathcal{F}_0) - \sigma_n^2\|_{p/2} < \infty\,. \tag{3.2}$$

Then the series $\sum_{k\in\mathbb{Z}} \mathrm{Cov}(X_0, X_k)$ converges to some nonnegative σ^2, and

1. *$\zeta_r(P_{n^{-1/2}S_n}, G_{\sigma^2}) = O(n^{1-p/2})$ for $r \in [p-2, 2]$,*
2. *$\zeta_r(P_{n^{-1/2}S_n}, G_{\sigma_n^2}) = O(n^{1-p/2})$ for $r \in]2, p]$.*

Remark 3.1. According to the bound (5.35), we infer that, under the assumptions of Theorem 3.1, the condition (3.2) is equivalent to

$$\sum_{n\ge 1} n^{-2+p/2} \| \, n^{-1}\,\mathbb{E}(S_n^2|\mathcal{F}_0) - \sigma^2\|_{p/2} < \infty\,. \tag{3.3}$$

The same remark applies to the next theorem with $p = 3$.

Remark 3.2. The result of item 1 is valid with σ_n instead of σ. On the contrary, the result of item 2 is no longer true if σ_n is replaced by σ, because for $r \in]2, 3]$, a necessary condition for $\zeta_r(\mu,\nu)$ to be finite is that the two first moments of ν and μ are equal. Note that under the

assumptions of Theorem 3.1, both $W_r(P_{n^{-1/2}S_n}, G_{\sigma^2})$ and $W_r(P_{n^{-1/2}S_n}, G_{\sigma_n^2})$ are of the order of $n^{-(p-2)/2\max(1,r)}$. Indeed, in the case where $r \in]2, p]$, one has that

$$W_r(P_{n^{-1/2}S_n}, G_{\sigma^2}) \leq W_r(P_{n^{-1/2}S_n}, G_{\sigma_n^2}) + W_r(G_{\sigma_n^2}, G_{\sigma^2}),$$

and the second term is of order $|\sigma - \sigma_n| = O(n^{-1/2})$.

In the case where $p = 3$, the condition (3.1) has to be strengthened.

Theorem 3.2. *Let $(X_i)_{i\in\mathbb{Z}}$ be a stationary sequence of centered random variables in $\mathbb{L}^3$, and let $\sigma_n^2 = n^{-1}\mathbb{E}(S_n^2)$. Assume that*

$$\sum_{n\geq 1} \frac{1}{n} \Big\| \sum_{k\geq n} \mathbb{E}(X_k|\mathcal{F}_0) \Big\|_3 < \infty \quad \textit{and} \quad \sum_{n\geq 1} \frac{1}{n} \Big\| \sum_{k\geq n} (X_{-k} - \mathbb{E}(X_{-k}|\mathcal{F}_0)) \Big\|_3 < \infty . \tag{3.4}$$

Assume in addition that

$$\sum_{n\geq 1} n^{-1/2} \| \, n^{-1} \, \mathbb{E}(S_n^2|\mathcal{F}_0) - \sigma_n^2 \|_{3/2} < \infty . \tag{3.5}$$

Then the series $\sum_{k\in\mathbb{Z}} \mathrm{Cov}(X_0, X_k)$ converges to some nonnegative σ^2 and

1. $\zeta_1(P_{n^{-1/2}S_n}, G_{\sigma^2}) = O(n^{-1/2}\log n)$,
2. $\zeta_r(P_{n^{-1/2}S_n}, G_{\sigma^2}) = O(n^{-1/2})$ *for* $r \in]1, 2]$,
3. $\zeta_r(P_{n^{-1/2}S_n}, G_{\sigma_n^2}) = O(n^{-1/2})$ *for* $r \in]2, 3]$.

4 Applications

4.1 Martingale differences sequences and functions of Markov chains

Recall that the strong mixing coefficient of Rosenblatt (1956) between two σ-algebras $\mathcal{A}$ and $\mathcal{B}$ is defined by $\alpha(\mathcal{A}, \mathcal{B}) = \sup\{|\mathbb{P}(A\cap B) - \mathbb{P}(A)\mathbb{P}(B)| \, : \, (A, B) \in \mathcal{A}\times\mathcal{B}\}$. For a strictly stationary sequence $(X_i)_{i\in\mathbb{Z}}$, let $\mathcal{F}_i = \sigma(X_k, k \leq i)$. Define the mixing coefficients $\alpha_1(n)$ of the sequence $(X_i)_{i\in\mathbb{Z}}$ by

$$\alpha_1(n) = \alpha(\mathcal{F}_0, \sigma(X_n)) .$$

Let Q be the quantile function of $|X_0|$, that is the cadlag inverse of the tail function $x \to \mathbb{P}(|X_0| > x)$. According to the results of Section 2, the following proposition holds.

Proposition 4.1. *Let $(X_i)_{i\in\mathbb{Z}}$ be a stationary martingale difference sequence. Assume moreover that the series*

$$\sum_{k\geq 1}\frac{1}{k^{2-p/2}}\int_0^{\alpha_1(k)}(1\vee\log(1/u))^{(p-2)/2}Q^2(u)du \quad \textit{and} \quad \sum_{k\geq 1}\frac{1}{k^{2/p}}\Big(\int_0^{\alpha_1(k)}Q^p(u)du\Big)^{2/p} \tag{4.1}$$

are convergent. Then the conclusions of Theorem 2.1 hold.

Remark 4.1. From Theorem 2.1(b) in Dedecker and Rio (2007), a sufficient condition to get $W_1(P_{n^{-1/2}S_n}, G_{\sigma^2}) = O(n^{-1/2}\log n)$ is

$$\sum_{k\geq 0}\int_0^{\alpha_1(n)}Q^3(u)du<\infty\,.$$

This condition is always strictly stronger than the condition (4.1) when $p=3$.

We now give an example. Consider the homogeneous Markov chain $(Y_i)_{i\in\mathbb{Z}}$ with state space $\mathbb{Z}$ described at page 320 in Davydov (1973). The transition probabilities are given by $p_{n,n+1} = p_{-n,-n-1} = a_n$ for $n\geq 0$, $p_{n,0} = p_{-n,0} = 1-a_n$ for $n>0$, $p_{0,0}=0$, $a_0 = 1/2$ and $1/2\leq a_n<1$ for $n\geq 1$. This chain is irreducible and aperiodic. It is Harris positively recurrent as soon as $\sum_{n\geq 2}\Pi_{k=1}^{n-1}a_k<\infty$. In that case the stationary chain is strongly mixing in the sense of Rosenblatt (1956).

Denote by K the Markov kernel of the chain $(Y_i)_{i\in\mathbb{Z}}$. The functions f such that $K(f)=0$ almost everywhere are obtained by linear combinations of the two functions f_1 and f_2 given by $f_1(1)=1$, $f_1(-1)=-1$ and $f_1(n)=f_1(-n)=0$ if $n\neq 1$, and $f_2(0)=1$, $f_2(1)=f_2(-1)=0$ and $f_2(n+1)=f_2(-n-1)=1-a_n^{-1}$ if $n>0$. Hence the functions f such that $K(f)=0$ are bounded.

If $(X_i)_{i\in\mathbb{Z}}$ is defined by $X_i=f(Y_i)$, with $K(f)=0$, then Proposition 4.1 applies if

$$\alpha_1(n)=O(n^{1-p/2}(\log n)^{-p/2-\epsilon}) \text{ for some } \epsilon>0, \tag{4.2}$$

which holds as soon as $P_0(\tau=n)=O(n^{-1-p/2}(\log n)^{-p/2-\epsilon})$, where P_0 is the probability of the chain starting from 0, and $\tau=\inf\{n>0, X_n=0\}$. Now $P_0(\tau=n)=(1-a_n)\Pi_{i=1}^{n-1}a_i$ for $n\geq 2$. Consequently, if

$$a_i=1-\frac{p}{2i}\Big(1+\frac{1+\epsilon}{\log i}\Big) \text{ for } i \text{ large enough}\,,$$

the condition (4.2) is satisfied and the conclusion of Theorem 2.1 holds.

Remark 4.2. If f is bounded and $K(f)\neq 0$, the central limit theorem may fail to hold for $S_n=\sum_{i=1}^n(f(Y_i)-\mathbb{E}(f(Y_i)))$. We refer to the Example 2, page 321, given by Davydov (1973), where S_n properly normalized converges to a stable law with exponent strictly less than 2.

Proof of Proposition 4.1. Let $B^p(\mathcal{F}_0)$ be the set of $\mathcal{F}_0$-measurable random variables such that $\|Z\|_p \leq 1$. We first notice that

$$\|\mathbb{E}(X_k^2|\mathcal{F}_0) - \sigma^2\|_{p/2} = \sup_{Z \in B^{p/(p-2)}(\mathcal{F}_0)} \mathrm{Cov}(Z, X_k^2)\,.$$

Applying Rio's covariance inequality (1993), we get that

$$\|\mathbb{E}(X_k^2|\mathcal{F}_0) - \sigma^2\|_{p/2} \leq 2\Big(\int_0^{\alpha_1(k)} Q^p(u)du\Big)^{2/p},$$

which shows that the convergence of the second series in (4.1) implies (2.2). Now, from Fréchet (1957), we have that

$$\|\mathbb{E}(X_k^2|\mathcal{F}_0) - \sigma^2\|_{1,\Phi,p} = \sup\big\{\mathbb{E}((1 \vee |Z|^{p-2})|\,\mathbb{E}(X_k^2|\mathcal{F}_0) - \sigma^2|\,), Z\ \mathcal{F}_0\text{-measurable},\ Z \sim \mathcal{N}(0,1)\big\}\,.$$

Hence, setting $\varepsilon_k = \mathrm{sign}(\mathbb{E}(X_k^2|\mathcal{F}_0) - \sigma^2)$,

$$\|\mathbb{E}(X_k^2|\mathcal{F}_0) - \sigma^2\|_{1,\Phi,p} = \sup\big\{\mathrm{Cov}(\varepsilon_k(1 \vee |Z|^{p-2}), X_k^2), Z\ \mathcal{F}_0\text{-measurable},\ Z \sim \mathcal{N}(0,1)\big\}\,.$$

Applying again Rio's covariance inequality (1993), we get that

$$\|\mathbb{E}(X_k^2|\mathcal{F}_0) - \sigma^2\|_{1,\Phi,p} \leq C\Big(\int_0^{\alpha_1(k)} (1 \vee \log(u^{-1}))^{(p-2)/2} Q^2(u)du\Big)\,,$$

which shows that the convergence of the first series in (4.1) implies (2.1).

4.2 Linear processes and functions of linear processes

Theorem 4.1. *Let $(a_i)_{i\in\mathbb{Z}}$ be a sequence of real numbers in ℓ^2 such that $\sum_{i\in\mathbb{Z}} a_i$ converges to some real A. Let $(\varepsilon_i)_{i\in\mathbb{Z}}$ be a stationary sequence of martingale differences in $\mathbb{L}^p$ for $p \in]2,3]$. Let $X_k = \sum_{j\in\mathbb{Z}} a_j\varepsilon_{k-j}$, and $\sigma_n^2 = n^{-1}\mathbb{E}(S_n^2)$. Let $b_0 = a_0 - A$ and $b_j = a_j$ for $j \neq 0$. Let $A_n = \sum_{j\in\mathbb{Z}}(\sum_{k=1}^n b_{k-j})^2$. If $A_n = o(n)$, then σ_n^2 converges to $\sigma^2 = A^2\mathbb{E}(\varepsilon_0^2)$. If moreover*

$$\sum_{n=1}^{\infty} \frac{1}{n^{2-p/2}} \Big\| \mathbb{E}\Big(\frac{1}{n}\Big(\sum_{j=1}^n \varepsilon_j\Big)^2 \Big| \mathcal{F}_0\Big) - \mathbb{E}(\varepsilon_0^2)\Big\|_{p/2} < \infty\,, \tag{4.3}$$

then we have

1. *If $A_n = O(1)$, then $\zeta_1(P_{n^{-1/2}S_n}, G_{\sigma^2}) = O(n^{-1/2}\log(n))$, for $p = 3$,*
2. *If $A_n = O(n^{(r+2-p)/r})$, then $\zeta_r(P_{n^{-1/2}S_n}, G_{\sigma^2}) = O(n^{1-p/2})$, for $r \in [p-2, 1]$ and $p \neq 3$,*
3. *If $A_n = O(n^{3-p})$, then $\zeta_r(P_{n^{-1/2}S_n}, G_{\sigma^2}) = O(n^{1-p/2})$, for $r \in]1, 2]$,*

4. If $A_n = O(n^{3-p})$, then $\zeta_r(P_{n^{-1/2}S_n}, G_{\sigma_n^2}) = O(n^{1-p/2})$, for $r \in]2, p]$.

Remark 4.3. If the condition given by Heyde (1975) holds, that is

$$\sum_{n=1}^{\infty} \Big(\sum_{k \geq n} a_k \Big)^2 < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} \Big(\sum_{k \leq -n} a_k \Big)^2 < \infty \,, \tag{4.4}$$

then $A_n = O(1)$, so that it satisfies all the conditions of items 1-4. On the other and, one has the bound

$$A_n \leq 4B_n, \quad \text{where} \quad B_n = \sum_{k=1}^{n} \Big(\Big(\sum_{j \geq k} |a_j| \Big)^2 + \Big(\sum_{j \leq -k} |a_j| \Big)^2 \Big) \,. \tag{4.5}$$

Proof of Theorem 4.1. We start with the following decomposition:

$$S_n = A \sum_{j=1}^{n} \varepsilon_j + \sum_{j=-\infty}^{\infty} \Big(\sum_{k=1}^{n} b_{k-j} \Big) \varepsilon_j \,. \tag{4.6}$$

Let $R_n = \sum_{j=-\infty}^{\infty} (\sum_{k=1}^{n} b_{k-j}) \varepsilon_j$. Since $\|R_n\|_2^2 = A_n \|\varepsilon_0\|_2^2$ and since $|\sigma_n - \sigma| \leq n^{-1/2} \|R_n\|_2$, the fact that $A_n = o(n)$ implies that σ_n converges to σ. We now give an upper bound for $\|R_n\|_p$. From Burkholder's inequality, there exists a constant C such that

$$\|R_n\|_p \leq C \Big\{ \Big\| \sum_{j=-\infty}^{\infty} \Big(\sum_{k=1}^{n} b_{k-j} \Big)^2 \varepsilon_j^2 \Big\|_{p/2} \Big\}^{1/2} \leq C \|\varepsilon_0\|_p \sqrt{A_n}. \tag{4.7}$$

The result follows by applying Theorem 2.1 to the martingale $A \sum_{k=1}^{n} \varepsilon_k$ (this is possible because of (4.3)), and by using Lemma 5.2 with the upper bound (4.7). To prove Remark 4.3, note first that

$$A_n = \sum_{j=1}^{n} \Big(\sum_{l=-\infty}^{-j} a_l + \sum_{l=n+1-j}^{\infty} a_l \Big)^2 + \sum_{i=1}^{\infty} \Big(\sum_{l=i}^{n+i-1} a_l \Big)^2 + \sum_{i=1}^{\infty} \Big(\sum_{l=-i-n+1}^{-i} a_l \Big)^2 .$$

It follows easily that $A_n = O(1)$ under (4.4). To prove the bound (4.5), note first that

$$A_n \leq 3B_n + \sum_{i=n+1}^{\infty} \Big(\sum_{l=i}^{n+i-1} |a_l| \Big)^2 + \sum_{i=n+1}^{\infty} \Big(\sum_{l=-i-n+1}^{-i} |a_l| \Big)^2 .$$

Let $T_i = \sum_{l=i}^{\infty} |a_l|$ and $Q_i = \sum_{l=-\infty}^{-i} |a_l|$. We have that

$$\begin{aligned} \sum_{i=n+1}^{\infty} \Big(\sum_{l=i}^{n+i-1} |a_l| \Big)^2 &\leq T_{n+1} \sum_{i=n+1}^{\infty} (T_i - T_{n+i}) \leq n T_{n+1}^2 \\ \sum_{i=n+1}^{\infty} \Big(\sum_{l=-i-n+1}^{-i} |a_l| \Big)^2 &\leq Q_{n+1} \sum_{i=n+1}^{\infty} (Q_i - Q_{n+i}) \leq n Q_{n+1}^2 . \end{aligned}$$

Since $n(T_{n+1}^2 + Q_{n+1}^2) \le B_n$, (4.5) follows. $\square$

In the next result, we shall focus on functions of real-valued linear processes

$$X_k = h\Big(\sum_{i\in\mathbb{Z}} a_i\varepsilon_{k-i}\Big) - \mathbb{E}\Big(h\Big(\sum_{i\in\mathbb{Z}} a_i\varepsilon_{k-i}\Big)\Big)\,, \tag{4.8}$$

where $(\varepsilon_i)_{i\in\mathbb{Z}}$ is a sequence of iid random variables. Denote by $w_h(.,M)$ the modulus of continuity of the function h on the interval $[-M,M]$, that is

$$w_h(t,M) = \sup\{|h(x)-h(y)|, |x-y|\le t, |x|\le M, |y|\le M\}\,.$$

Theorem 4.2. *Let $(a_i)_{i\in\mathbb{Z}}$ be a sequence of real numbers in ℓ^2 and $(\varepsilon_i)_{i\in\mathbb{Z}}$ be a sequence of iid random variables in $\mathbb{L}^2$. Let X_k be defined as in (4.8) and $\sigma_n^2 = n^{-1}\mathbb{E}(S_n^2)$. Assume that h is γ-Hölder on any compact set, with $w_h(t,M)\le Ct^\gamma M^\alpha$, for some $C>0$, $\gamma\in]0,1]$ and $\alpha\ge 0$. If for some $p\in]2,3]$,*

$$\mathbb{E}(|\varepsilon_0|^{2\vee(\alpha+\gamma)p}) < \infty \quad \textit{and} \quad \sum_{i\ge 1} i^{p/2-1}\Big(\sum_{|j|\ge i} a_j^2\Big)^{\gamma/2} < \infty, \tag{4.9}$$

then the series $\sum_{k\in\mathbb{Z}}\mathrm{Cov}(X_0,X_k)$ converges to some nonnegative σ^2, and

1. *$\zeta_1(P_{n^{-1/2}S_n}, G_{\sigma^2}) = O(n^{-1/2}\log n)$, for $p=3$,*
2. *$\zeta_r(P_{n^{-1/2}S_n}, G_{\sigma^2}) = O(n^{1-p/2})$ for $r\in[p-2,2]$ and $(r,p)\ne(1,3)$,*
3. *$\zeta_r(P_{n^{-1/2}S_n}, G_{\sigma_n^2}) = O(n^{1-p/2})$ for $r\in]2,p]$.*

Proof of Theorem 4.2. Theorem 4.2 is a consequence of the following proposition:

Proposition 4.2. *Let $(a_i)_{i\in\mathbb{Z}}$, $(\varepsilon_i)_{i\in\mathbb{Z}}$ and $(X_i)_{i\in\mathbb{Z}}$ be as in Theorem 4.2. Let $(\varepsilon_i')_{i\in\mathbb{Z}}$ be an independent copy of $(\varepsilon_i)_{i\in\mathbb{Z}}$. Let $V_0 = \sum_{i\in\mathbb{Z}} a_i\varepsilon_{-i}$ and*

$$M_{1,i} = |V_0| \vee \Big|\sum_{j<i} a_j\varepsilon_{-j} + \sum_{j\ge i} a_j\varepsilon'_{-j}\Big| \quad \textit{and} \quad M_{2,i} = |V_0| \vee \Big|\sum_{j<i} a_j\varepsilon'_{-j} + \sum_{j\ge i} a_j\varepsilon_{-j}\Big|.$$

If for some $p\in]2,3]$,

$$\sum_{i\ge 1} i^{p/2-1}\Big\|w_h\Big(\Big|\sum_{j\ge i} a_j\varepsilon_{-j}\Big|, M_{1,i}\Big)\Big\|_p < \infty \quad \textit{and} \quad \sum_{i\ge 1} i^{p/2-1}\Big\|w_h\Big(\Big|\sum_{j<-i} a_j\varepsilon_{-j}\Big|, M_{2,-i}\Big)\Big\|_p < \infty, \tag{4.10}$$

then the conclusions of Theorem 4.2 hold.

To prove Theorem 4.2, it remains to check (4.10). We only check the first condition. Since $w_h(t,M) \le Ct^\gamma M^\alpha$ and the random variables ε_i are iid, we have

$$\Big\| w_h\Big(\Big|\sum_{j\ge i} a_j\varepsilon_{-j}\Big|, M_{1,i}\Big)\Big\|_p \;\le\; C\Big\|\Big|\sum_{j\ge i} a_j\varepsilon_{-j}\Big|^\gamma |V_0|^\alpha\Big\|_p + C\Big\|\Big|\sum_{j\ge i} a_j\varepsilon_{-j}\Big|^\gamma\Big\|_p \||V_0|^\alpha\|_p\,,$$

so that

$$\begin{aligned}&\Big\| w_h\Big(\Big|\sum_{j\ge i} a_j\varepsilon_{-j}\Big|, M_{1,i}\Big)\Big\|_p \\ &\qquad\le C\Big(2^\alpha\Big\|\Big|\sum_{j\ge i} a_j\varepsilon_{-j}\Big|^{\alpha+\gamma}\Big\|_p + \Big\|\Big|\sum_{j\ge i} a_j\varepsilon_{-j}\Big|^\gamma\Big\|_p\Big(\||V_0|^\alpha\|_p + 2^\alpha\Big\|\Big|\sum_{j<i} a_j\varepsilon_{-j}\Big|^\alpha\Big\|_p\Big)\Big).\end{aligned}$$

From Burkholder's inequality, for any $\beta > 0$,

$$\Big\|\Big|\sum_{j\ge i} a_j\varepsilon_{-j}\Big|^\beta\Big\|_p = \Big\|\sum_{j\ge i} a_j\varepsilon_{-j}\Big\|_{\beta p}^\beta \le K\Big(\sum_{j\ge i} a_j^2\Big)^{\beta/2}\|\varepsilon_0\|_{2\vee\beta p}^\beta\,.$$

Applying this inequality with $\beta = \gamma$ or $\beta = \alpha+\gamma$, we infer that the first part of (4.10) holds under (4.9). The second part can be handled in the same way. □

Proof of Proposition 4.2. Let $\mathcal{F}_i = \sigma(\varepsilon_k, k \le i)$. We shall first prove that the condition (3.2) of Theorem 3.1 holds. We write

$$\begin{aligned}\|\mathbb{E}(S_n^2|\mathcal{F}_0) \;-\; \mathbb{E}(S_n^2)\|_{p/2} &\le 2\sum_{i=1}^n\sum_{k=0}^{n-i}\|\mathbb{E}(X_iX_{k+i}|\mathcal{F}_0) - \mathbb{E}(X_iX_{k+i})\|_{p/2} \\ &\le\; 4\sum_{i=1}^n\sum_{k=i}^n\|\mathbb{E}(X_iX_{k+i}|\mathcal{F}_0)\|_{p/2} + 2\sum_{i=1}^n\sum_{k=1}^i\|\mathbb{E}(X_iX_{k+i}|\mathcal{F}_0) - \mathbb{E}(X_iX_{k+i})\|_{p/2}\,.\end{aligned}$$

We first control the second term. Let ε' be an independent copy of ε, and denote by $\mathbb{E}_\varepsilon(\cdot)$ the conditional expectation with respect to ε. Define

$$Y_i = \sum_{j<i} a_j\varepsilon_{i-j}\;,\; Y_i' = \sum_{j<i} a_j\varepsilon'_{i-j}\;, Z_i = \sum_{j\ge i} a_j\varepsilon_{i-j}\;,\; Z_i' = \sum_{j\ge i} a_j\varepsilon'_{i-j}$$

and $m_{1,i} = |Y_i' + Z_i| \vee |Y_i' + Z_i'|$. Taking $\mathcal{F}_\ell = \sigma(\varepsilon_i, i \le \ell)$, and setting $h_0 = h - \mathbb{E}(h(\sum_{i\in\mathbb{Z}} a_i\varepsilon_i))$, we have

$$\begin{aligned}&\|\mathbb{E}(X_iX_{k+i}|\mathcal{F}_0) - \mathbb{E}(X_iX_{k+i})\|_{p/2} \\ &\qquad= \Big\|\mathbb{E}_\varepsilon\Big(h_0(Y_i' + Z_i)h_0(Y'_{k+i} + Z_{k+i})\Big) - \mathbb{E}_\varepsilon\Big(h_0(Y_i' + Z_i')h_0(Y'_{k+i} + Z'_{k+i})\Big)\Big\|_{p/2}\,.\end{aligned}$$

Hence,

$$\begin{aligned}\|\mathbb{E}(X_iX_{k+i}|\mathcal{F}_0)-\mathbb{E}(X_iX_{k+i})\|_{p/2} &\leq \|h_0(Y'_{k+i}+Z_{k+i})\|_p\Big\|w_h\Big(\Big|\sum_{j\geq i}a_j(\varepsilon_{i-j}-\varepsilon'_{i-j})\Big|,m_{1,i}\Big)\Big\|_p \\ &\quad+\|h_0(Y'_i+Z'_i)\|_p\Big\|w_h\Big(\Big|\sum_{j\geq k+i}a_j(\varepsilon_{k+i-j}-\varepsilon'_{k+i-j})\Big|,m_{1,k+i}\Big)\Big\|_p .\end{aligned}$$

By subadditivity,

$$\begin{aligned}\Big\|w_h\Big(\Big|\sum_{j\geq i}a_j(\varepsilon_{i-j}-\varepsilon'_{i-j})\Big|,m_{1,i}\Big)\Big\|_p &\leq \Big\|w_h\Big(\Big|\sum_{j\geq i}a_j\varepsilon_{i-j}\Big|,m_{1,i}\Big)\Big\|_p+\Big\|w_h\Big(\Big|\sum_{j\geq i}a_j\varepsilon'_{i-j}\Big|,m_{1,i}\Big)\Big\|_p \\ &\leq 2\Big\|w_h\Big(\Big|\sum_{j\geq i}a_j\varepsilon_{-j}\Big|,M_{1,i}\Big)\Big\|_p .\end{aligned}$$

In the same way

$$\Big\|w_h\Big(\Big|\sum_{j\geq k+i}a_j(\varepsilon_{k+i-j}-\varepsilon'_{k+i-j})\Big|,m_{1,k+i}\Big)\Big\|_p\leq 2\Big\|w_h\Big(\Big|\sum_{j\geq k+i}a_j\varepsilon_{-j}\Big|,M_{1,k+i}\Big)\Big\|_p .$$

Consequently

$$\sum_{n\geq 1}\frac{1}{n^{3-p/2}}\sum_{i=1}^n\sum_{k=1}^i\|\mathbb{E}(X_iX_{k+i}|\mathcal{F}_0)-\mathbb{E}(X_iX_{k+i})\|_{p/2}<\infty$$

provided that the first condition in (4.10) holds.

We turn now to the control of $\sum_{i=1}^n\sum_{k=i}^n\|\mathbb{E}(X_iX_{k+i}|\mathcal{F}_0)\|_{p/2}$. We first write that

$$\begin{aligned}\|\mathbb{E}(X_iX_{k+i}|\mathcal{F}_0)\|_{p/2} &= \|\mathbb{E}\big((X_i-\mathbb{E}(X_i|\mathcal{F}_{i+[k/2]}))X_{k+i}|\mathcal{F}_0\big)\|_{p/2}+\|\mathbb{E}\big(\mathbb{E}(X_i|\mathcal{F}_{i+[k/2]})X_{k+i}|\mathcal{F}_0\big)\|_{p/2} \\ &= \|X_0\|_p\|X_i-\mathbb{E}(X_i|\mathcal{F}_{i+[k/2]})\|_p+\|X_0\|_p\|\mathbb{E}(X_{k+i}|\mathcal{F}_{i+[k/2]})\|_p .\end{aligned}$$

Let $b(k)=k-[k/2]$. Since $\|\mathbb{E}(X_{k+i}|\mathcal{F}_{i+[k/2]})\|_p=\|\mathbb{E}(X_{b(k)}|\mathcal{F}_0)\|_p$, we have that

$$\begin{aligned}&\|\mathbb{E}(X_{k+i}|\mathcal{F}_{i+[k/2]})\|_p \\ &\quad=\Big\|\mathbb{E}_\varepsilon\Big(h\Big(\sum_{j<b(k)}a_j\varepsilon'_{b(k)-j}+\sum_{j\geq b(k)}a_j\varepsilon_{b(k)-j}\Big)-h\Big(\sum_{j<b(k)}a_j\varepsilon'_{b(k)-j}+\sum_{j\geq b(k)}a_j\varepsilon'_{b(k)-j}\Big)\Big)\Big\|_p .\end{aligned}$$

Using the same arguments as before, we get that

$$\|\mathbb{E}(X_{k+i}|\mathcal{F}_{i+[k/2]})\|_p\leq 2\Big\|w_h\Big(\Big|\sum_{j\geq b(k)}a_j\varepsilon_{-j}\Big|,M_{1,b(k)}\Big)\Big\|_p .$$

In the same way,

$$\begin{aligned}&\Big\|X_i-\mathbb{E}(X_i|\mathcal{F}_{i+[k/2]})\Big\|_p \\ &\quad=\Big\|\mathbb{E}_\varepsilon\Big(h\Big(\sum_{j<-[k/2]}a_j\varepsilon_{i-j}+\sum_{j\geq -[k/2]}a_j\varepsilon_{i-j}\Big)-h\Big(\sum_{j<-[k/2]}a_j\varepsilon'_{i-j}+\sum_{j\geq -[k/2]}a_j\varepsilon_{i-j}\Big)\Big)\Big\|_p ,\end{aligned}$$

so that

$$\Big\|X_i - \mathbb{E}(X_i|\mathcal{F}_{i+[k/2]})\Big\|_p \le 2\Big\|w_h\Big(\Big|\sum_{j<-[k/2]} a_j\varepsilon_{-j}\Big|, M_{2,-[k/2]}\Big)\Big\|_p .$$

Consequently

$$\sum_{n\ge1}\frac{1}{n^{3-p/2}}\sum_{i=1}^{n}\sum_{k=i}^{n}\|\mathbb{E}(X_iX_{k+i}|\mathcal{F}_0)\|_{p/2}<\infty$$

provided that (4.10) holds. This completes the proof of (3.2). Using the same arguments, one can easily check that the condition (3.1) of Theorem 3.1 (and also the condition (3.4) of Theorem 3.2 in the case $p=3$) holds under (4.10). □

4.3 Functions of ϕ-dependent sequences

In order to include examples of dynamical systems satisfying some correlations inequalities, we introduce a weak version of the uniform mixing coefficients (see Dedecker and Prieur (2007)).

Definition 4.1. For any random variable $Y=(Y_1,\cdots,Y_k)$ with values in $\mathbb{R}^k$ define the function $g_{x,j}(t)=\mathbb{1}_{t\le x}-\mathbb{P}(Y_j\le x)$. For any σ-algebra $\mathcal{F}$, let

$$\phi(\mathcal{F},Y)=\sup_{(x_1,\dots,x_k)\in\mathbb{R}^k}\Big\|\mathbb{E}\Big(\prod_{j=1}^{k}g_{x_j,j}(Y_j)\Big|\mathcal{F}\Big)-\mathbb{E}\Big(\prod_{j=1}^{k}g_{x_j,j}(Y_j)\Big)\Big\|_\infty .$$

For a sequence $\mathbf{Y}=(Y_i)_{i\in\mathbb{Z}}$, where $Y_i=Y_0\circ T^i$ and Y_0 is a $\mathcal{F}_0$-measurable and real-valued r.v., let

$$\phi_{k,\mathbf{Y}}(n)=\max_{1\le l\le k}\sup_{i_l>\dots>i_1\ge n}\phi(\mathcal{F}_0,(Y_{i_1},\dots,Y_{i_l})).$$

Definition 4.2. For any $p\ge1$, let $\mathcal{C}(p,M,P_X)$ be the closed convex envelop of the set of functions f which are monotonous on some open interval of $\mathbb{R}$ and null elsewhere, and such that $\mathbb{E}(|f(X)|^p)<M$.

Proposition 4.3. *Let $p\in]2,3]$ and $s\ge p$. Let $X_i=f(Y_i)-\mathbb{E}(f(Y_i))$, where $Y_i=Y_0\circ T^i$ and f belongs to $\mathcal{C}(s,M,P_{Y_0})$. Assume that*

$$\sum_{i\ge1} i^{(p-4)/2+(s-2)/(s-1)}\phi_{2,\mathbf{Y}}(i)^{(s-2)/s}<\infty\,. \tag{4.11}$$

Then the conclusions of Theorem 4.2 hold.

Remark 4.4. Notice that if $s=p=3$, the condition (4.11) becomes $\sum_{i\ge1}\phi_{2,\mathbf{Y}}(i)^{1/3}<\infty$, and if $s=\infty$, the condition (4.11) becomes $\sum_{i\ge1} i^{(p-2)/2}\phi_{2,\mathbf{Y}}(i)<\infty$.

Proof of Proposition 4.3. Let $B^p(\mathcal{F}_0)$ be the set of $\mathcal{F}_0$-measurable random variables such that $\|Z\|_p \leq 1$. We first notice that

$$\|\mathbb{E}(X_k|\mathcal{F}_0)\|_p \leq \|\mathbb{E}(X_k|\mathcal{F}_0)\|_s = \sup_{Z \in B^{s/(s-1)}(\mathcal{F}_0)} \mathrm{Cov}(Z, f(Y_k))\,.$$

According to Corollary 6.2 and since $\phi(\sigma(Z), Y_k) \leq \phi_{1,\mathbf{Y}}(k)$, we get that

$$\|\mathbb{E}(X_k|\mathcal{F}_0)\|_s \leq 8M^{1/s}(\phi_{1,\mathbf{Y}}(k))^{(s-1)/s}\,. \tag{4.12}$$

It follows that the conditions (3.1) (for $p \in]2,3[$) or (3.4) (for $p=3$) are satisfied under (4.11). The condition (3.2) follows from the following lemma by taking $b = (4-p)/2$.

Lemma 4.1. *Let X_i be as in Proposition 4.3, and let $b \in]0,1[$.*

$$\textit{If} \quad \sum_{i\geq 1} i^{-b+(s-2)/(s-1)}\phi_{2,\mathbf{Y}}(i)^{(s-2)/s} < \infty, \quad \textit{then} \quad \sum_{n>1}\frac{1}{n^{1+b}}\|\mathbb{E}(S_n^2|\mathcal{F}_0) - \mathbb{E}(S_n^2)\|_{p/2} < \infty\,.$$

Proof of Lemma 4.1. Since,

$$\|\mathbb{E}(S_n^2|\mathcal{F}_0) - \mathbb{E}(S_n^2)\|_{p/2} \leq 2\sum_{i=1}^{n}\sum_{k=0}^{n-i}\|\mathbb{E}(X_iX_{k+i}|\mathcal{F}_0) - \mathbb{E}(X_iX_{k+i})\|_{p/2},$$

we infer that there exists $C>0$ such that

$$\sum_{n>1}\frac{1}{n^{1+b}}\|\mathbb{E}(S_n^2|\mathcal{F}_0) - \mathbb{E}(S_n^2)\|_{p/2} \leq C\sum_{i>0}\sum_{k\geq 0}\frac{1}{(i+k)^b}\|\mathbb{E}(X_iX_{k+i}|\mathcal{F}_0) - \mathbb{E}(X_iX_{k+i})\|_{p/2}\,. \tag{4.13}$$

We shall bound up $\|\mathbb{E}(X_iX_{k+i}|\mathcal{F}_0) - \mathbb{E}(X_iX_{k+i})\|_{p/2}$ in two ways. First, using the stationarity and the upper bound (4.12), we have that

$$\|\mathbb{E}(X_iX_{k+i}|\mathcal{F}_0) - \mathbb{E}(X_iX_{k+i})\|_{p/2} \leq 2\|X_0\mathbb{E}(X_k|\mathcal{F}_0)\|_{p/2} \leq 16\|X_0\|_p M^{1/s}(\phi_{1,\mathbf{Y}}(k))^{(s-1)/s}\,.$$

Next, using again Corollary 6.2,

$$\|\mathbb{E}(X_iX_{k+i}|\mathcal{F}_0) - \mathbb{E}(X_iX_{k+i})\|_{p/2} \leq \sup_{Z\in B^{s/(s-2)}(\mathcal{F}_0)}\mathrm{Cov}(Z, X_iX_{k+i}) \leq 32M^{2/s}(\phi_{2,\mathbf{Y}}(i))^{(s-2)/s}\,.$$

From (4.13) and the above upper bounds, we infer that the conclusion of Lemma (4.1) holds provided that

$$\sum_{i>0}\Big(\sum_{k=1}^{[i^{(s-2)/(s-1)}]}\frac{1}{(i+k)^b}\Big)(\phi_{2,\mathbf{Y}}(i))^{(s-2)/s} + \sum_{k\geq 0}\Big(\sum_{i=1}^{[k^{(s-1)/(s-2)}]}\frac{1}{(i+k)^b}\Big)(\phi_{1,\mathbf{Y}}(k))^{(s-1)/s} < \infty\,.$$

Here, note that

$$\sum_{k=1}^{[i^{(s-2)/(s-1)}]} \frac{1}{(i+k)^b} \leq i^{-b+\frac{s-2}{s-1}} \quad \text{and} \quad \sum_{i=1}^{[k^{(s-1)/(s-2)}]} \frac{1}{(i+k)^b} \leq \sum_{m=1}^{[2k^{(s-1)/(s-2)}]} \frac{1}{m^b} \leq Dk^{(1-b)\frac{(s-1)}{(s-2)}},$$

for some $D > 0$. Since $\phi_{1,\mathbf{Y}}(k) \leq \phi_{2,\mathbf{Y}}(k)$, the conclusion of lemma (4.1) holds provided

$$\sum_{i\geq 1} i^{-b+\frac{s-2}{s-1}} \phi_{2,\mathbf{Y}}(i)^{\frac{s-2}{s}} < \infty \quad \text{and} \quad \sum_{k\geq 1} k^{(1-b)\frac{(s-1)}{(s-2)}} \phi_{2,\mathbf{Y}}(k)^{\frac{s-1}{s}} < \infty .$$

One can prove that the second series converges provided the first one does. □

4.3.1 Application to Expanding maps

Let BV be the class of bounded variation functions from $[0,1]$ to $\mathbb{R}$. For any $h \in BV$, denote by $\|dh\|$ the variation norm of the measure dh.

Let T be a map from $[0,1]$ to $[0,1]$ preserving a probability μ on $[0,1]$, and let

$$S_n(f) = \sum_{k=1}^{n} (f \circ T^k - \mu(f)) .$$

Define the Perron-Frobenius operator K from $\mathbb{L}^2([0,1],\mu)$ to $\mathbb{L}^2([0,1],\mu)$ *via* the equality

$$\int_0^1 (Kh)(x) f(x) \mu(dx) = \int_0^1 h(x)(f \circ T)(x) \mu(dx) . \tag{4.14}$$

A Markov Kernel K is said to be BV-contracting if there exist $C > 0$ and $\rho \in [0,1[$ such that

$$\|dK^n(h)\| \leq C\rho^n \|dh\| . \tag{4.15}$$

The map T is said to be BV-contracting if its Perron-Frobenius operator is BV-contracting.

Let us present a large class of BV-contracting maps. We shall say that T is uniformly expanding if it belongs to the class $\mathcal{C}$ defined in Broise (1996), Section 2.1 page 11. Recall that if T is uniformly expanding, then there exists a probability measure μ on $[0,1]$, whose density f_μ with respect to the Lebesgue measure is a bounded variation function, and such that μ is invariant by T. Consider now the more restrictive conditions:

(a) T is uniformly expanding.

(b) The invariant measure μ is unique and (T,μ) is mixing in the ergodic-theoretic sense.

(c) $\frac{1}{f_\mu} \mathbf{1}_{f_\mu > 0}$ is a bounded variation function.

Starting from Proposition 4.11 in Broise (1996), one can prove that if T satisfies the assumptions (a), (b) and (c) above, then it is BV contracting (see for instance Dedecker and Prieur (2007), Section 6.3). Some well known examples of maps satisfying the conditions (a), (b) and (c) are:

1. $T(x) = \beta x - [\beta x]$ for $\beta > 1$. These maps are called β-transformations.
2. I is the finite union of disjoint intervals $(I_k)_{1\le k\le n}$, and $T(x) = a_k x + b_k$ on I_k, with $|a_k| > 1$.
3. $T(x) = a(x^{-1} - 1) - [a(x^{-1} - 1)]$ for some $a > 0$. For $a = 1$, this transformation is known as the Gauss map.

Proposition 4.4. *Let $\sigma_n^2 = n^{-1}\mathbb{E}(S_n^2(f))$. If T is BV-contracting, and if f belongs to $\mathcal{C}(p, M, \mu)$ with $p \in]2, 3]$, then the series $\mu((f - \mu(f))^2) + 2\sum_{n>0}\mu(f \circ T^n \cdot (f - \mu(f)))$ converges to some nonnegative σ^2, and*

1. *$\zeta_1(P_{n^{-1/2}S_n(f)}, G_{\sigma^2}) = O(n^{-1/2}\log n)$, for $p = 3$,*
2. *$\zeta_r(P_{n^{-1/2}S_n(f)}, G_{\sigma^2}) = O(n^{1-p/2})$ for $r \in [p-2, 2]$ and $(r, p) \neq (1, 3)$,*
3. *$\zeta_r(P_{n^{-1/2}S_n(f)}, G_{\sigma_n^2}) = O(n^{1-p/2})$ for $r \in]2, p]$.*

Proof of Proposition 4.4. Let $(Y_i)_{i\ge1}$ be the Markov chain with transition Kernel K and invariant measure μ. Using the equation (4.14) it is easy to see that $(Y_0, \dots, Y_n)$ it is distributed as $(T^{n+1}, \dots, T)$. Consequently, to prove Proposition 4.4, it suffices to prove that the sequence $X_i = f(Y_i) - \mu(f)$ satisfies the condition (4.11) of Proposition 4.3.

According to Lemma 1 in Dedecker and Prieur (2007), the coefficients $\phi_{2,\mathbf{Y}}(i)$ of the chain $(Y_i)_{i\ge0}$ with respect to $\mathcal{F}_i = \sigma(Y_j, j \le i)$ satisfy $\phi_{2,\mathbf{Y}}(i) \le C\rho^i$ for some $\rho \in]0, 1[$ and some positive constant C. It follows that (4.11) is satisfied for $s = p$.

5 Proofs of the main results

From now on, we denote by C a numerical constant which may vary from line to line.

Notation 5.1. For l integer, q in $]l, l+1]$ and f l-times continuously differentiable, we set

$$|f|_{\Lambda_q} = \sup\{|x - y|^{l-q}|f^{(l)}(x) - f^{(l)}(y)| : (x, y) \in \mathbb{R} \times \mathbb{R}\}.$$

5.1 Proof of Theorem 2.1

We prove Theorem 2.1 in the case $\sigma = 1$. The general case follows by dividing the random variables by σ. Since $\zeta_r(P_{aX}, P_{aY}) = |a|^r \zeta_r(P_X, P_Y)$, it is enough to bound up $\zeta_r(P_{S_n}, G_n)$. We first give an upper bound for $\zeta_{p,N} := \zeta_p(P_{S_{2^N}}, G_{2^N})$.

Proposition 5.1. *Let $(X_i)_{i\in\mathbb{Z}}$ be a stationary martingale differences sequence. Let $M_p = \mathbb{E}(|X_0|^p)$. Then for any p in $]2,3]$ and any natural integer N,*

$$2^{-2N/p}\zeta_{p,N}^{2/p} \le \Big(M_p + \frac{1}{2\sqrt{2}}\sum_{K=0}^{N} 2^{K(p/2-2)}\|Z_K\|_{1,\Phi,p}\Big)^{2/p} + \frac{2}{p}\Delta_N\,, \tag{5.1}$$

where $Z_K = \mathbb{E}(S_{2^K}^2|\mathcal{F}_0) - \mathbb{E}(S_{2^K}^2)$ and $\Delta_N = \sum_{K=0}^{N-1} 2^{-2K/p}\|Z_K\|_{p/2}$.

Proof of Proposition 5.1. The proof is done by induction on N. Let $(Y_i)_{i\in\mathbb{N}}$ be a sequence of $N(0,1)$-distributed independent random variables, independent of the sequence $(X_i)_{i\in\mathbb{Z}}$. For $m > 0$, let $T_m = Y_1 + Y_2 + \cdots + Y_m$. Set $S_0 = T_0 = 0$. For f numerical function and $m \le n$, set

$$f_{n-m}(x) = \mathbb{E}(f(x + T_n - T_m)).$$

Then, from the independence of the above sequences,

$$\mathbb{E}(f(S_n) - f(T_n)) = \sum_{m=1}^{n} D_m \quad \text{with} \quad D_m = \mathbb{E}\big(f_{n-m}(S_{m-1} + X_m) - f_{n-m}(S_{m-1} + Y_m)\big). \tag{5.2}$$

Next, from the Taylor integral formula at order two, for any two-times differentiable function g and any q in $]2,3]$,

$$\begin{aligned} |g(x+h) - g(x) - g'(x)h - \tfrac{1}{2}h^2 g''(x)| &\le h^2 \int_0^1 (1-t)|g''(x+th) - g''(x)|dt \\ &\le h^2 \int_0^1 (1-t)|th|^{q-2}|g|_{\Lambda_q} dt, \end{aligned}$$

whence

$$|g(x+h) - g(x) - g'(x)h - \frac{1}{2}h^2 g''(x)| \le \frac{1}{q(q-1)}|h|^q |g|_{\Lambda_q}. \tag{5.3}$$

Let

$$D'_m = \mathbb{E}(f''_{n-m}(S_{m-1})(X_m^2 - 1)) = \mathbb{E}(f''_{n-m}(S_{m-1})(X_m^2 - Y_m^2))$$

From (5.3) applied twice with $g = f_{n-m}$, $x = S_{m-1}$ and $h = X_m$ or $h = Y_m$ together with the martingale property,

$$\Big|D_m - \frac{1}{2}D'_m\Big| \le \frac{1}{p(p-1)}\,|f_{n-m}|_{\Lambda_p}\mathbb{E}(|X_m|^p + |Y_m|^p).$$

Now $\mathbb{E}(|Y_m|^p) \leq p-1 \leq (p-1)M_p$. Hence

$$|D_m - (D'_m/2)| \leq M_p |f_{n-m}|_{\Lambda_p} \tag{5.4}$$

Moreover, if f belongs to Λ_p, then the smoothed function f_{n-m} belongs to Λ_p. Hence, summing on m, we get that

$$\mathbb{E}(f(S_n) - f(T_n)) \leq nM_p + (D'/2) \quad \text{where } D' = D'_1 + D'_2 + \cdots + D'_n. \tag{5.5}$$

Suppose now that $n = 2^N$. To bound up D', we introduce a dyadic scheme.

Notation 5.2. Set $m_0 = m-1$ and write m_0 in basis 2: $m_0 = \sum_{i=0}^N b_i 2^i$ with $b_i = 0$ or $b_i = 1$ (note that $b_N = 0$). Set $m_L = \sum_{i=L}^N b_i 2^i$, so that $m_N = 0$. Let $I_{L,k} =]k2^L, (k+1)2^L] \cap \mathbb{N}$ (note that $I_{N,1} =]2^N, 2^{N+1}]$), $U_L^{(k)} = \sum_{i \in I_{L,k}} X_i$ and $\tilde{U}_L^{(k)} = \sum_{i \in I_{L,k}} Y_i$. For the sake of brevity, let $U_L^{(0)} = U_L$ and $\tilde{U}_L^{(0)} = \tilde{U}_L$.

Since $m_N = 0$, the following elementary identity is valid

$$D'_m = \sum_{L=0}^{N-1} \mathbb{E}\Big((f''_{n-1-m_L}(S_{m_L}) - f''_{n-1-m_{L+1}}(S_{m_{L+1}}))(X_m^2 - 1) \Big).$$

Now $m_L \neq m_{L+1}$ only if $b_L = 1$, then in this case $m_L = k2^L$ with k odd. It follows that

$$D' = \sum_{L=0}^{N-1} \sum_{\substack{k \in I_{N-L,0} \\ k \text{ odd}}} \mathbb{E}\Big((f''_{n-1-k2^L}(S_{k2^L}) - f''_{n-1-(k-1)2^L}(S_{(k-1)2^L})) \sum_{\{m : m_L = k2^L\}} (X_m^2 - \sigma^2) \Big). \tag{5.6}$$

Note that $\{m : m_L = k2^L\} = I_{L,k}$. Now by the martingale property

$$\mathbb{E}_{k2^L}\Big(\sum_{i \in I_{L,k}} (X_i^2 - \sigma^2) \Big) = \mathbb{E}_{k2^L}((U_L^{(k)})^2) - \mathbb{E}((U_L^{(k)})^2) := Z_L^{(k)}\,.$$

Since $(X_i)_{i \in \mathbb{N}}$ and $(Y_i)_{i \in \mathbb{N}}$ are independent, we infer that

$$D' = \sum_{L=0}^{N-1} \sum_{\substack{k \in I_{N-L,0} \\ k \text{ odd}}} \mathbb{E}\Big(\big(f''_{n-1-k2^L}(S_{k2^L}) - f''_{n-1-k2^L}(S_{(k-1)2^L} + T_{k2^L} - T_{(k-1)2^L}) \big) Z_L^{(k)} \Big). \tag{5.7}$$

By using (1.2), we get that

$$D' \leq \sum_{L=0}^{N-1} \sum_{\substack{k \in I_{N-L,0} \\ k \text{ odd}}} \mathbb{E}(|U_L^{(k-1)} - \tilde{U}_L^{(k-1)}|^{p-2} |Z_L^{(k)}|)\,.$$

From the stationarity of $(X_i)_{i\in\mathbb{N}}$ and the above inequality,

$$D' \leq \frac{1}{2}\sum_{K=0}^{N-1} 2^{N-K}\mathbb{E}(|U_K - \tilde{U}_K|^{p-2}|Z_K^{(1)}|). \tag{5.8}$$

Now let V_K be the $N(0,2^K)$-distributed random variable defined from U_K via the quantile transformation, that is

$$V_K = 2^{K/2}\Phi^{-1}(F_K(U_K - 0) + \delta_K(F_K(U_K) - F_K(U_K - 0)))$$

where F_K denotes the d.f. of U_K, and (δ_K) is a sequence of independent uniformly distributed r.v.'s, independent of the underlying random variables. Now, from the subadditivity of $x \to x^{p-2}$, $|U_K - \tilde{U}_K|^{p-2} \leq |U_K - V_K|^{p-2} + |V_K - \tilde{U}_K|^{p-2}$. Hence

$$\mathbb{E}(|U_K - \tilde{U}_K|^{p-2}|Z_K^{(1)}|) \leq \|U_K - V_K\|_p^{p-2}\|Z_K^{(1)}\|_{p/2} + \mathbb{E}(|V_K - \tilde{U}_K|^{p-2}|Z_K^{(1)}|)\,. \tag{5.9}$$

By definition of V_K, the real $\|U_K - V_K\|_p$ is the so-called Wasserstein distance of order p between the law of $U_K^{(0)}$ and the $N(0,2^K)$ normal law. Therefrom, by Theorem 3.1 of Rio (2007) (which improves the constants given in Theorem 1 of Rio (1998)), we get that

$$\|U_K - V_K\|_p \leq 2(2(p-1)\zeta_{p,K})^{1/p}. \tag{5.10}$$

Now, since V_K and $\tilde{U}_K$ are independent, their difference has the $N(0,2^{K+1})$ distribution. Hence, by definition of the envelope norm $\|\,.\,\|_{1,\Phi,p}$,

$$\mathbb{E}(|V_K - \tilde{U}_K|^{p-2}|Z_K^{(1)}|) \leq 2^{(K+1)(p/2-1)}\|Z_K\|_{1,\Phi,p}. \tag{5.11}$$

From (5.9), (5.10) and (5.11), we get that

$$\mathbb{E}(|U_K - \tilde{U}_K|^{p-2}|Z_K^{(1)}|) \leq 2^{p-4/p}\zeta_{p,K}^{\frac{p-2}{p}}\|Z_K\|_{p/2} + 2^{(K+1)(p/2-1)}\|Z_K\|_{1,\Phi,p}\,. \tag{5.12}$$

Then, from (5.5), (5.8) and (5.12), we get

$$2^{-N}\zeta_{p,N} \leq M_p + 2^{p/2-3}\Delta'_N + 2^{p-2-4/p}\sum_{K=0}^{N-1} 2^{-K}\zeta_{p,K}^{\frac{p-2}{p}}\|Z_K\|_{p/2},$$

where $\Delta'_N = \sum_{K=0}^{N-1} 2^{K(p/2-2)}\|Z_K\|_{1,\Phi,p}$. Consequently we get the induction inequality

$$2^{-N}\zeta_{p,N} \leq M_p + \frac{1}{2\sqrt{2}}\Delta'_N + \sum_{K=0}^{N-1} 2^{-K}\zeta_{p,K}^{\frac{p-2}{p}}\|Z_K\|_{p/2}\,. \tag{5.13}$$

We now prove (5.1) by induction on N. Assume that $\zeta_{p,L}$ satisfies (5.1) for any L in $[0, N-1]$. Starting from (5.13), using the induction hypothesis and the fact that $\Delta'_K \leq \Delta'_N$, we get that

$$2^{-N}\zeta_{p,N} \leq M_p + \frac{1}{2\sqrt{2}}\Delta'_N + \sum_{K=0}^{N-1} 2^{-2K/p}\|Z_K\|_{p/2}\Big(\Big(M_p + \frac{1}{2\sqrt{2}}\Delta'_N\Big)^{2/p} + \frac{2}{p}\Delta_K\Big)^{p/2-1}.$$

Now $2^{-2K/p}\|Z_K\|_{p/2} = \Delta_{K+1} - \Delta_K$. Consequently

$$2^{-N}\zeta_{p,N} \leq M_p + \frac{1}{2\sqrt{2}}\Delta'_N + \int_0^{\Delta_N} \Big(\Big(M_p + \frac{1}{2\sqrt{2}}\Delta'_N\Big)^{2/p} + \frac{2}{p}x\Big)^{p/2-1} dx\,,$$

which implies (5.1) for $\zeta_{p,N}$. □

In order to prove Theorem 2.1, we will also need a smoothing argument. This is the purpose of the lemma below.

Lemma 5.1. *For any r in $]0, p]$, $\zeta_r(P_{S_n}, G_n) \leq 2\zeta_r(P_{S_n} * G_1, G_n * G_1) + 4\sqrt{2}$.*

Proof of Lemma 5.1. Throughout the sequel, let Y be a $N(0,1)$-distributed random variable, independent of the σ-field generated by the random variables $(X_i)_i$ and $(Y_i)_i$.

For $r \leq 2$, since ζ_r is an ideal metric with respect to the convolution,

$$\zeta_r(P_{S_n}, G_n) \leq \zeta_r(P_{S_n} * G_1, G_n * G_1) + 2\zeta_r(\delta_0, G_1) \leq \zeta_r(P_{S_n} * G_1, G_n * G_1) + 2\mathbb{E}|Y|^r$$

which implies Lemma 5.1 for $r \leq 2$. For $r > 2$, from (5.3), for any f in Λ_r,

$$f(S_n) - f(S_n + Y) + f'(S_n)Y - \tfrac{1}{2}f''(S_n)Y^2 \leq \tfrac{1}{r(r-1)}|Y|^r.$$

Taking the expectation and noting that $\mathbb{E}|Y|^r \leq r - 1$ for r in $]2, 3]$, we infer that

$$\mathbb{E}(f(S_n) - f(S_n + Y) - \tfrac{1}{2}f''(S_n)) \leq \tfrac{1}{r}.$$

Obviously this inequality still holds for T_n instead of S_n and $-f$ instead of f, so that adding the so obtained inequality,

$$\mathbb{E}(f(S_n) - f(T_n) \leq \mathbb{E}(f(S_n + Y) - f(T_n + Y)) + \tfrac{1}{2}\mathbb{E}(f''(S_n) - f''(T_n)) + 1.$$

It follows that

$$\zeta_r(P_{S_n}, G_n) \leq \zeta_r(P_{S_n} * G_1, G_n * G_1) + \tfrac{1}{2}\zeta_{r-2}(P_{S_n}, G_n) + 1.$$

Now $r - 2 \leq 1$. Hence

$$\zeta_{r-2}(P_{S_n}, G_n) = W_{r-2}((P_{S_n}, G_n) \leq (W_r(P_{S_n}, G_n))^{r-2}.$$

Next, by Theorem 3.1 in Rio (2007), $W_r(P_{S_n}, G_n) \le (32\zeta_r(P_{S_n}, G_n))^{1/r}$. Furthermore

$$(32\zeta_r(P_{S_n}, G_n))^{1-2/r} \le \zeta_r(P_{S_n}, G_n)$$

as soon as $\zeta_r(P_{S_n}, G_n) \ge 2^{(5r/2)-5}$. This condition holds for any r in $]2, 3]$ if $\zeta_r(P_{S_n}, G_n) \ge 4\sqrt{2}$. Then, from the above inequalities

$$\zeta_r(P_{S_n}, G_n) \le \zeta_r(P_{S_n} * G_1, G_n * G_1) + \tfrac{1}{2}\zeta_r(P_{S_n}, G_n) + 1,$$

which implies Lemma 5.1 □

We go back to the proof of Theorem 2.1. We will first complete the proof in the case $p = r$. Next we will derive the general case at the end of the proof.

Let $\zeta^*_{p,N} = \sup_{n \le 2^N} \zeta_p(P_{S_n}, G_n)$. We will bound up $\zeta^*_{p,N}$ by induction on N. Let $n \in]2^N, 2^{N+1}]$. Hence $n = 2^N + \ell$ with $\ell \in [1, 2^N]$. We first notice that

$$\zeta_r(P_{S_n}, G_n) \le \zeta_r(P_{S_n}, P_{S_\ell} * G_{2^N}) + \zeta_r(P_{S_\ell} * G_{2^N}, G_\ell * G_{2^N})\,.$$

Now, with the same notation as in the proof of Proposition 5.1, we have

$$\zeta_r(P_{S_\ell} * G_{2^N}, G_\ell * G_{2^N}) = \sup_{f \in \Lambda_r} \mathbb{E}(f_{2^N}(S_\ell) - f_{2^N}(T_\ell)) \le |f * \phi_{2^{N/2}}|_{\Lambda_p} \zeta_p(P_{S_\ell}, G_\ell)\,.$$

Applying Lemma 6.1, we infer that

$$\zeta_r(P_{S_n}, G_n) \le \zeta_r(P_{S_n}, P_{S_\ell} * G_{2^N}) + c_{r,p} 2^{N(r-p)/2} \zeta_p(P_{S_\ell}, G_\ell)\,. \tag{5.14}$$

On the other hand, setting $\tilde{S}_\ell = X_{1-\ell} + \cdots + X_0$, we have that S_n is distributed as $\tilde{S}_\ell + S_{2^N}$. Using Lemma 5.1, we then derive that

$$\zeta_r(P_{S_n}, P_{S_\ell} * G_{2^N}) \le 4\sqrt{2} + 2 \sup_{f \in \Lambda_r} \mathbb{E}(f(\tilde{S}_\ell + S_{2^N} + Y) - f(\tilde{S}_\ell + T_{2^N} + Y)) \tag{5.15}$$

Let $D'_m = \mathbb{E}(f''_{2^N - m+1}(\tilde{S}_\ell + S_{m-1})(X_m^2 - 1))$. Following the proof of Proposition 5.1, we get that

$$\mathbb{E}(f(\tilde{S}_\ell + S_{2^N} + Y) - f(\tilde{S}_\ell + T_{2^N} + Y)) = (D'_1 + \cdots + D'_{2^N})/2 + R_1 + \cdots + R_{2^N}\,, \tag{5.16}$$

where, as in (5.4),

$$R_m \le M_p |f_{2^N - m+1}|_{\Lambda_p}\,. \tag{5.17}$$

In the case $r = p - 2$, we will need the more precise upper bound

$$R_m \le \mathbb{E}\Big(X_m^2 \big(\|f''_{2^N - m+1}\|_\infty \wedge \frac{1}{6}\|f^{(3)}_{2^N - m+1}\|_\infty |X_m|\big)\Big) + \frac{1}{6}\|f^{(3)}_{2^N - m+1}\|_\infty \mathbb{E}(|Y_m|^3)\,, \tag{5.18}$$

which is derived from the Taylor formula at orders two and three. From (5.17) and Lemma 6.1, we have that

$$R := R_1 + \cdots + R_{2^N} = O(2^{N(r-p+2)/2}) \quad \text{if } r > p-2, \text{ and} \quad R = O(N) \quad \text{if } (r,p) = (1,3)\,. \tag{5.19}$$

It remains to consider the case $r = p-2$ and $r < 1$. Applying Lemma 6.1, we get that for $i \geq 2$,

$$\|f^{(i)}_{2^N-m+1}\|_\infty \leq c_{r,i}(2^N - m + 1)^{(r-i)/2}\,. \tag{5.20}$$

It follows that

$$\begin{aligned}\sum_{m=1}^{2^N} \mathbb{E}\Big(X_m^2\big(\|f''_{2^N-m+1}\|_\infty \wedge \|f^{(3)}_{2^N-m+1}\|_\infty |X_m|\big)\Big) &\leq C\sum_{m=1}^{\infty} \frac{1}{m^{1-r/2}}\mathbb{E}\left(X_0^2\Big(1 \wedge \frac{|X_0|}{\sqrt{m}}\Big)\right) \\ &\leq C\mathbb{E}\Big(\sum_{m=1}^{[X_0^2]} \frac{X_0^2}{m^{1-r/2}} + \sum_{m=[X_0^2]+1}^{\infty} \frac{|X_0|^3}{m^{(3-r)/2}}\Big)\,.\end{aligned}$$

Consequently for $r = p-2$ and $r < 1$,

$$R_1 + \cdots + R_{2^N} \leq C(M_p + \mathbb{E}(|Y|^3))\,. \tag{5.21}$$

We now bound up $D'_1 + \cdots + D'_{2^N}$. Using the dyadic scheme as in the proof of Proposition 5.1, we get that

$$\begin{aligned} D'_m &= \sum_{L=0}^{N-1} \mathbb{E}\Big((f''_{2^N-m_L}(\tilde{S}_\ell + S_{m_L}) - f''_{2^N-m_{L+1}}(\tilde{S}_\ell + S_{m_{L+1}})(X_m^2 - 1)\Big) + \mathbb{E}(f''_{2^N}(\tilde{S}_\ell)(X_m^2 - 1)) \\ &:= D''_m + \mathbb{E}(f''_{2^N}(\tilde{S}_\ell)(X_m^2 - 1))\,.\end{aligned}$$

Notice first that

$$\sum_{m=1}^{2^N} \mathbb{E}(f''_{2^N}(\tilde{S}_\ell)(X_m^2 - 1)) = \mathbb{E}((f''_{2^N}(\tilde{S}_\ell) - f''_{2^N}(T_\ell))Z_N^{(0)})\,.$$

Hence using Lemma 6.1, we get that

$$\sum_{m=1}^{2^N} \mathbb{E}(f''_{2^N}(\tilde{S}_\ell)(X_m^2 - 1)) \leq C2^{N(r-p)/2}\mathbb{E}(|\tilde{S}_\ell - T_\ell|^{p-2}|Z_N^{(0)}|)\,.$$

Proceeding as to get (5.12), we have that

$$\mathbb{E}(|\tilde{S}_\ell - T_\ell|^{p-2}|Z_N^{(0)}|) \leq 2^{p-4/p}(\zeta_p(P_{S_\ell}, G_\ell))^{(p-2)/p}\|Z_N^{(0)}\|_{p/2} + (2\ell)^{p/2-1}\|Z_N^{(0)}\|_{1,\Phi,p}\,.$$

Using Remark 2.5, (2.1) and (2.2) entail that $\|Z_N^{(0)}\|_{p/2} = o(2^{2N/p})$ and $\|Z_N^{(0)}\|_{1,\Phi,p} = o(2^{N(2-p/2)})$. Hence, for some $\epsilon(N)$ tending to 0 as N tends to infinity, one has

$$D'_1 + \cdots + D'_{2^N} \le C(\epsilon(N)2^{N((r-p)/2+2/p}(\zeta_p(P_{S_\ell}, G_\ell))^{(p-2)/p} + 2^{N(r+2-p)/2}) \,. \tag{5.22}$$

Next, proceeding as in the proof of (5.7), we get that

$$\sum_{m=1}^{2^N} D''_m \le \sum_{L=0}^{N-1} \sum_{\substack{k\in I_{N-L,0}\\ k \text{ odd}}} \mathbb{E}\Big(\big(f''_{2^N-k2^L}(\tilde{S}_\ell + S_{k2^L}) - f''_{2^N-k2^L}(\tilde{S}_\ell + S_{(k-1)2^L} + T_{k2^L} - T_{(k-1)2^L})\big) Z_L^{(k)} \Big) \,.$$

If $r > p-2$ or $(r,p) = (1,3)$, from Lemma 6.1, the stationarity of $(X_i)_{i\in\mathbb{N}}$ and the above inequality,

$$\sum_{m=1}^{2^N} D''_m \;\le\; C \sum_{L=0}^{N-1} \sum_{\substack{k\in I_{N-L,0}\\ k \text{ odd}}} (2^N - k2^L)^{(r-p)/2} \mathbb{E}\big(|U_L - \tilde{U}_L|^{p-2} \big| Z_L^{(1)}\big|\big) \,.$$

It follows that

$$\sum_{m=1}^{2^N} D''_m \;\le\; C2^{N(r+2-p)/2} \sum_{L=0}^{N} 2^{-L} \mathbb{E}\big(\big|U_L - \tilde{U}_L\big|^{p-2} \big| Z_L^{(1)}\big|\big) \quad \text{if } r > p-2, \tag{5.23}$$

$$\sum_{m=1}^{2^N} D''_m \;\le\; CN \sum_{L=0}^{N} 2^{-L} \mathbb{E}\big(\big|U_L - \tilde{U}_L\big| \big| Z_L^{(1)}\big|\big) \quad \text{if } r = 1 \text{ and } p = 3. \tag{5.24}$$

In the case $r = p-2$ and $r > 1$, we have

$$\sum_{m=1}^{2^N} D''_m \;\le\; C \sum_{L=0}^{N-1} \sum_{\substack{k\in I_{N-L,0}\\ k \text{ odd}}} \mathbb{E}\Big(\big(\|f''_{2^N-k2^L}\|_\infty \wedge \|f'''_{2^N-k2^L}\|_\infty \big|U_L - \tilde{U}_L\big|\big) \big| Z_L^{(1)}\big| \Big) \,.$$

Applying (5.20) to $i = 2$ and $i = 3$, we obtain

$$\sum_{m=1}^{2^N} D''_m \le C \sum_{L=0}^{N} 2^{(r-2)L/2} \mathbb{E}\Big(\big| Z_L^{(1)}\big| \sum_{k=1}^{2^{N-L}} k^{(r-2)/2} \big(1 \wedge \frac{1}{2^{L/2}\sqrt{k}} \big|U_L - \tilde{U}_L\big|\big) \Big) \,,$$

Proceeding as to get (5.21), we have that

$$\sum_{k=1}^{2^{N-L}} k^{(r-2)/2} \big(1 \wedge \frac{1}{2^{L/2}\sqrt{k}} \big|U_L - \tilde{U}_L\big|\big) \le \sum_{k=1}^{\infty} k^{(r-2)/2} \big(1 \wedge \frac{1}{2^{L/2}\sqrt{k}} \big|U_L - \tilde{U}_L\big|\big) \le C |U_L - \tilde{U}_L|^r \,.$$

It follows that

$$\sum_{m=1}^{2^N} D''_m \le C \sum_{L=0}^{N} 2^{-L} \mathbb{E}\Big(\big|U_L - \tilde{U}_L\big|^r \big| Z_L^{(1)}\big| \Big) \quad \text{if } r = p-2 \text{ and } r < 1. \tag{5.25}$$

Now by Remark 2.5, (2.1) and (2.2) are respectively equivalent to

$$\sum_{K\geq 0} 2^{K(p/2-2)}\|Z_K\|_{1,\Phi,p} < \infty\,, \text{ and } \sum_{K\geq 0} 2^{-2K/p}\|Z_K\|_{p/2} < \infty\,.$$

Next, by Proposition 5.1, $\zeta_{p,K} = O(2^K)$ under (2.1) and (2.2). Therefrom, taking into account the inequality (5.12), we derive that under (2.1) and (2.2),

$$2^{-L}\mathbb{E}\Big(\big|U_L - \tilde{U}_L\big|^{p-2}\big|Z_L^{(1)}\big|\Big) \;\leq\; C2^{-2L/p}\|Z_L\|_{p/2} + C2^{L(p/2-2)}\|Z_K\|_{1,\Phi,p}\,. \tag{5.26}$$

Consequently, combining (5.26) with the upper bounds (5.23), (5.24) and (5.25), we obtain that

$$\sum_{m=1}^{2^N} D''_m = \begin{cases} O(2^{N(r+2-p)/2}) & \text{if } r \geq p-2 \text{ and } (r,p) \neq (1,3) \\ O(N) & \text{if } r = 1 \text{ and } p = 3. \end{cases} \tag{5.27}$$

From (5.14), (5.15), (5.16), (5.19), (5.21), (5.22) and (5.27), we get that if $r \geq p-2$ and $(r,p) \neq (1,3)$,

$$\zeta_r(P_{S_n}, G_n) \leq c_{r,p}2^{N(r-p)/2}\zeta_p(P_{S_\ell}, G_\ell) + C(2^{N(r+2-p)/2} + 2^{N((r-p)/2+2/p)}\epsilon(N)(\zeta_p(P_{S_\ell}, G_\ell))^{(p-2)/p}) \tag{5.28}$$

and if $r = 1$ and $p = 3$,

$$\zeta_1(P_{S_n}, G_n) \leq C(N + 2^{-N}\zeta_3(P_{S_\ell}, G_\ell) + 2^{-N/3}(\zeta_3(P_{S_\ell}, G_\ell))^{1/3})\,. \tag{5.29}$$

Since $\zeta^*_{p,N} = \sup_{n\leq 2^N}\zeta_p(P_{S_n}, G_n)$, we infer from (5.28) applied to $r = p$ that

$$\zeta^*_{p,N+1} \leq \zeta^*_{p,N} + C(2^N + 2^{2N/p}\epsilon(N)(\zeta^*_{p,N})^{(p-2)/p})\,.$$

Let N_0 be such that $C\epsilon(N) \leq 1/2$ for $N \leq N_0$, and let $K \geq 1$ be such that $\zeta^*_{p,N_0} \leq K2^{N_0}$. Choosing K large enough such that $K \geq 2C$, we can easily prove by induction that $\zeta^*_{p,N} \leq K2^N$ for any $N \geq N_0$. Hence Theorem 2.1 is proved in the case $r = p$.

For r in $[p-2, p[$, Theorem 2.1 follows by taking into account the bound $\zeta^*_{p,N} \leq K2^N$, valid for any $N \geq N_0$, in the inequalities (5.28) and (5.29).

5.2 Proof of Theorem 3.1

By (3.1), we get that (see Volný (1993))

$$X_0 = D_0 + Z_0 - Z_0 \circ T, \tag{5.30}$$

where

$$Z_0 = \sum_{k=0}^{\infty}\mathbb{E}(X_k|\mathcal{F}_{-1}) - \sum_{k=1}^{\infty}(X_{-k} - \mathbb{E}(X_{-k}|\mathcal{F}_{-1})) \quad \text{and} \quad D_0 = \sum_{k\in\mathbb{Z}}\mathbb{E}(X_k|\mathcal{F}_0) - \mathbb{E}(X_k|\mathcal{F}_{-1})\,.$$

Note that $D_0 \in \mathbb{L}^p$, D_0 is $\mathcal{F}_0$-measurable, and $\mathbb{E}(D_0|\mathcal{F}_{-1}) = 0$. Let $D_i = D_0 \circ T^i$, and $Z_i = Z_0 \circ T^i$. We obtain that

$$S_n = M_n + Z_1 - Z_{n+1}\,, \tag{5.31}$$

where $M_n = \sum_{j=1}^n D_j$. We first bound up $\mathbb{E}(f(S_n) - f(M_n))$ by using the following lemma

Lemma 5.2. *Let $p \in]2,3]$ and $r \in [p-2, p]$. Let $(X_i)_{i\in\mathbb{Z}}$ be a stationary sequence of centered random variables in $\mathbb{L}^{2\vee r}$. Assume that $S_n = M_n + R_n$ where $(M_n - M_{n-1})_{n>1}$ is a strictly stationary sequence of martingale differences in $\mathbb{L}^{2\vee r}$, and R_n is such that $\mathbb{E}(R_n) = 0$. Let $n\sigma^2 = \mathbb{E}(M_n^2)$, $n\sigma_n^2 = \mathbb{E}(S_n^2)$ and $\alpha_n = \sigma_n/\sigma$.*

1. *If $r \in [p-2, 1]$ and $\mathbb{E}|R_n|^r = O(n^{(r+2-p)/2})$, then $\zeta_r(P_{S_n}, P_{M_n}) = O(n^{(r+2-p)/2})$.*
2. *If $r \in]1, 2]$ and $\|R_n\|_r = O(n^{(3-p)/2})$, then $\zeta_r(P_{S_n}, P_{M_n}) = O(n^{(r+2-p)/2})$.*
3. *If $r \in]2, p]$, $\sigma^2 > 0$ and $\|R_n\|_r = O(n^{(3-p)/2})$, then $\zeta_r(P_{S_n}, P_{\alpha_n M_n}) = O(n^{(r+2-p)/2})$.*
4. *If $r \in]2, p]$, $\sigma^2 = 0$ and $\|R_n\|_r = O(n^{(r+2-p)/2r})$, then $\zeta_r(P_{S_n}, G_{n\sigma_n^2}) = O(n^{(r+2-p)/2})$.*

Remark 5.1. All the assumptions of Lemma 5.2 are satisfied as soon as $\sup_{n>0} \|R_n\|_p < \infty$.

Proof of Lemma 5.2. For $r \in]0, 1]$, $\zeta_r(P_{S_n}, P_{M_n}) \le \mathbb{E}(|R_n|^r)$, which implies item 1. If $f \in \Lambda_r$ with $r \in]1, 2]$, from the Taylor integral formula and since $\mathbb{E}(R_n) = 0$, we get

$$\begin{aligned}
\mathbb{E}(f(S_n) - f(M_n)) &= \mathbb{E}\Big(R_n\Big(f'(M_n) - f'(0) + \int_0^1 (f'(M_n + t(R_n)) - f'(M_n))dt\Big)\Big) \\
&\le \|R_n\|_r \|f'(M_n) - f'(0)\|_{r/(r-1)} + \|R_n\|_r^r \le \|R_n\|_r \|M_n\|_r^{r-1} + \|R_n\|_r^r\,.
\end{aligned}$$

Since $\|M_n\|_r \le \|M_n\|_2 = \sqrt{n}\sigma$, we infer that $\zeta_r(P_{S_n}, P_{M_n}) = O(n^{(r+2-p)/2})$.

Now if $f \in \Lambda_r$ with $r \in]2, p]$ and if $\sigma > 0$, we define g by

$$g(t) = f(t) - tf'(0) - t^2 f''(0)/2\,.$$

The function g is then also in Λ_r and is such that $g'(0) = g''(0) = 0$. Since $\alpha_n^2 \mathbb{E}(M_n^2) = \mathbb{E}(S_n^2)$, we have

$$\mathbb{E}(f(S_n) - f(\alpha_n M_n)) = \mathbb{E}(g(S_n) - g(\alpha_n M_n))\,. \tag{5.32}$$

Now from the Taylor integral formula at order two, setting $\tilde{R}_n = R_n + (1 - \alpha_n)M_n$,

$$\begin{aligned}
\mathbb{E}(g(S_n) - g(\alpha_n M_n)) &= \mathbb{E}(\tilde{R}_n g'(\alpha_n M_n)) + \frac{1}{2}\mathbb{E}((\tilde{R}_n)^2 g''(\alpha_n M_n)) \\
&\quad + \mathbb{E}\big((\tilde{R}_n)^2 \int_0^1 (1-t)(g''(\alpha_n M_n + t\tilde{R}_n) - g''(\alpha_n M_n))dt\big) \\
&\le \frac{1}{r-1}\mathbb{E}(|\tilde{R}_n||\alpha_n M_n|^{r-1}) + \frac{1}{2}\|\tilde{R}_n\|_r^2 \|g''(\alpha_n M_n)\|_{r/(r-2)} + \frac{1}{2}\|\tilde{R}_n\|_r^r \\
&\le \frac{1}{r-1}\alpha_n^{r-1}\|\tilde{R}_n\|_r \|M_n\|_r^{r-1} + \frac{1}{2}\alpha_n^{r-2}\|\tilde{R}_n\|_r^2 \|M_n\|_r^{r-2} + \|\tilde{R}_n\|_r^r\,.
\end{aligned}$$

Now $\alpha_n = O(1)$ and $\|\tilde{R}_n\|_r \leq \|R_n\|_r + |1-\alpha_n|\|M_n\|_r$. Since $|\|S_n\|_2 - \|M_n\|_2| \leq \|R_n\|_2$, we infer that $|1-\alpha_n| = O(n^{(2-p)/2})$. Hence, applying Burkhölder's inequality for martingales, we infer that $\|\tilde{R}_n\|_r = O(n^{(3-p)/2})$, and consequently $\zeta_r(P_{S_n}, P_{\alpha_n M_n}) = O(n^{(r+2-p)/2})$.

If $\sigma^2 = 0$, then $S_n = R_n$. Using that $\mathbb{E}(f(S_n) - f(\sqrt{n}\sigma_n Y)) = \mathbb{E}(g(R_n) - g(\sqrt{n}\sigma_n Y))$, and applying again Taylor's formula, we obtain that

$$\sup_{f\in\Lambda_r} |\mathbb{E}(f(S_n) - f(\sqrt{n}\sigma_n Y))| \leq \frac{1}{r-1}\|\bar{R}_n\|_r \|\sqrt{n}\sigma_n Y\|_r^{r-1} + \frac{1}{2}\|\bar{R}_n\|_r^2 \|\sqrt{n}\sigma_n Y\|_r^{r-2} + \|\bar{R}_n\|_r^r \,,$$

where $\bar{R}_n = R_n - \sqrt{n}\sigma_n Y$. Since $\sqrt{n}\sigma_n = \|R_n\|_2 = O(n^{(r+2-p)/2r})$, the result follows. □

By (5.31), we can apply Lemma 5.2 with $R_n := Z_1 - Z_{n+1}$. Then for $p-2 \leq r \leq 2$, the result follows if we prove that under (3.2), M_n satisfies the conclusion of Theorem 2.1. Now if $2 < r \leq p$ and $\sigma^2 > 0$, we first notice that

$$\zeta_r(P_{\alpha_n M_n}, G_{n\sigma_n^2}) = \alpha_n^r \zeta_r(P_{M_n}, G_{n\sigma^2}) \,.$$

Since $\alpha_n = O(1)$, the result will follow by Item 3 of Lemma 5.2, if we prove that under (3.2), M_n satisfies the conclusion of Theorem 2.1. We shall prove that

$$\sum_{n\geq 1} \frac{1}{n^{3-p/2}} \|\mathbb{E}(M_n^2|\mathcal{F}_0) - \mathbb{E}(M_n^2)\|_{p/2} < \infty \,. \tag{5.33}$$

In this way, both (2.1) and (2.2) will be satisfied. Suppose that we can show that

$$\sum_{n\geq 1} \frac{1}{n^{3-p/2}} \|\mathbb{E}(M_n^2|\mathcal{F}_0) - \mathbb{E}(S_n^2|\mathcal{F}_0)\|_{p/2} < \infty \,, \tag{5.34}$$

then by taking into account the condition (3.2), (5.33) will follow. Indeed, it suffices to notice that (5.34) also entails that

$$\sum_{n\geq 1} \frac{1}{n^{3-p/2}} |\mathbb{E}(S_n^2) - \mathbb{E}(M_n^2)| < \infty \,, \tag{5.35}$$

and to write that

$$\begin{aligned} \|\mathbb{E}(M_n^2|\mathcal{F}_0) - \mathbb{E}(M_n^2)\|_{p/2} \;\leq\; & \|\mathbb{E}(M_n^2|\mathcal{F}_0) - \mathbb{E}(S_n^2|\mathcal{F}_0)\|_{p/2} \\ & + \|\mathbb{E}(S_n^2|\mathcal{F}_0) - \mathbb{E}(S_n^2)\|_{p/2} + |\mathbb{E}(S_n^2) - \mathbb{E}(M_n^2)| \,. \end{aligned}$$

Hence, it remains to prove (5.34). Since $S_n = M_n + Z_1 - Z_{n+1}$, and since $Z_i = Z_0 \circ T^i$ is in $\mathbb{L}^p$, (5.34) will be satisfied provided that

$$\sum_{n\geq 1} \frac{1}{n^{3-p/2}} \|S_n(Z_1 - Z_{n+1})\|_{p/2} < \infty \,. \tag{5.36}$$

Notice that

$$\|S_n(Z_1 - Z_{n+1})\|_{p/2} \;\leq\; \|M_n\|_p \|Z_1 - Z_{n+1}\|_p + \|Z_1 - Z_{n+1}\|_p^2 .$$

From Burkholder's inequality, $\|M_n\|_p = O(\sqrt{n})$ and from (3.1), $\sup_n \|Z_1 - Z_{n+1}\|_p < \infty$. Consequently (5.36) is satisfied for any p in $]2, 3[$.

5.3 Proof of Theorem 3.2

Starting from (5.31) we have that

$$M_n := S_n + R_n + \tilde{R}_n \,, \tag{5.37}$$

where

$$R_n = \sum_{k \geq n+1} \mathbb{E}(X_k|\mathcal{F}_n) - \sum_{k \geq 1} \mathbb{E}(X_k|\mathcal{F}_0) \text{ and } \tilde{R}_n = \sum_{k \geq 0} (X_{-k} - \mathbb{E}(X_{-k}|\mathcal{F}_0)) - \sum_{k \geq -n} (X_{-k} - \mathbb{E}(X_{-k}|\mathcal{F}_n)) \,.$$

Arguing as in the proof of Proposition 3.1 the proposition will follow from (3.5), if we prove that

$$\sum_{n \geq 1}^{\infty} \frac{1}{n^{3/2}} \|\mathbb{E}(M_n^2|\mathcal{F}_0) - \mathbb{E}(S_n^2|\mathcal{F}_0)\|_{3/2} < \infty \,. \tag{5.38}$$

Under (3.4), $\sup_{n \geq 1} \|R_n\|_3 < \infty$ and $\sup_{n \geq 1} \|\tilde{R}_n\|_3 < \infty$. Hence (5.38) will be verified as soon as

$$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \|\mathbb{E}(S_n(R_n + \tilde{R}_n)|\mathcal{F}_0)\|_{3/2} < \infty \,. \tag{5.39}$$

We first notice that the decomposition (5.37) together with Burkholder's inequality for martingales and the fact that $\sup_n \|R_n\|_3 < \infty$ and $\sup_n \|\tilde{R}_n\|_3 < \infty$, implies that

$$\|S_n\|_3 \leq C\sqrt{n} \,. \tag{5.40}$$

Now to prove (5.39), we first notice that

$$\Big\| \mathbb{E}\Big(S_n \sum_{k \geq 1} \mathbb{E}(X_k|\mathcal{F}_0) \Big| \mathcal{F}_0 \Big) \Big\|_{3/2} \leq \|\mathbb{E}(S_n|\mathcal{F}_0)\|_3 \Big\| \sum_{k \geq 1} \mathbb{E}(X_k|\mathcal{F}_0) \Big\|_3 \,, \tag{5.41}$$

which is bounded by using (3.4). Now write

$$\mathbb{E}\Big(S_n \sum_{k \geq n+1} \mathbb{E}(X_k|\mathcal{F}_n) \Big| \mathcal{F}_0 \Big) = \mathbb{E}\Big(S_n \sum_{k \geq 2n+1} \mathbb{E}(X_k|\mathcal{F}_n) \Big| \mathcal{F}_0 \Big) + \mathbb{E}(S_n \mathbb{E}(S_{2n} - S_n|\mathcal{F}_n)|\mathcal{F}_0) \,.$$

Clearly

$$\Big\|\mathbb{E}\Big(S_n \sum_{k\geq 2n+1} \mathbb{E}(X_k|\mathcal{F}_n)\Big|\mathcal{F}_0\Big)\Big\|_{3/2} \leq \|S_n\|_3 \Big\| \sum_{k\geq 2n+1} \mathbb{E}(X_k|\mathcal{F}_n)\Big\|_3$$
$$\leq C\sqrt{n}\Big\| \sum_{k\geq n+1} \mathbb{E}(X_k|\mathcal{F}_0)\Big\|_3 , \tag{5.42}$$

by using (5.40). Considering the bounds (5.41) and (5.42) and the condition (3.4), in order to prove that

$$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \|\mathbb{E}(S_n R_n|\mathcal{F}_0)\|_{3/2} < \infty , \tag{5.43}$$

it is sufficient to prove that

$$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \|\mathbb{E}(S_n \mathbb{E}(S_{2n} - S_n|\mathcal{F}_n)|\mathcal{F}_0)\|_{3/2} < \infty . \tag{5.44}$$

With this aim, take $p_n = [\sqrt{n}]$ and write

$$\mathbb{E}(S_n \mathbb{E}(S_{2n} - S_n|\mathcal{F}_n)|\mathcal{F}_0) = \mathbb{E}((S_n - S_{n-p_n})\mathbb{E}(S_{2n} - S_n|\mathcal{F}_n)|\mathcal{F}_0)$$
$$+\mathbb{E}(S_{n-p_n}\mathbb{E}(S_{2n} - S_n|\mathcal{F}_n)|\mathcal{F}_0). \tag{5.45}$$

By stationarity and (5.40), we get that

$$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \|\mathbb{E}((S_n - S_{n-p_n})\mathbb{E}(S_{2n} - S_n|\mathcal{F}_n)|\mathcal{F}_0)\|_{3/2} \leq C \sum_{n=1}^{\infty} \frac{\sqrt{p_n}}{n^{3/2}} \|\mathbb{E}(S_n|\mathcal{F}_0)\|_3 ,$$

which is finite under (3.4), since $p_n = [\sqrt{n}]$. Hence from (5.45), (5.44) will follow if we prove that

$$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \|\mathbb{E}(S_{n-p_n}\mathbb{E}(S_{2n} - S_n|\mathcal{F}_n)|\mathcal{F}_0)\|_{3/2} < \infty . \tag{5.46}$$

With this aim we first notice that

$$\|\mathbb{E}((S_{n-p_n} - \mathbb{E}(S_{n-p_n}|\mathcal{F}_{n-p_n})\mathbb{E}(S_{2n} - S_n|\mathcal{F}_n)|\mathcal{F}_0)\|_{3/2}$$
$$\leq \|S_{n-p_n} - \mathbb{E}(S_{n-p_n}|\mathcal{F}_{n-p_n})\|_3 \|\mathbb{E}(S_{2n} - S_n|\mathcal{F}_n)\|_3 ,$$

which is bounded under (3.4). Consequently (5.46) will hold if we prove that

$$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \|\mathbb{E}(\mathbb{E}(S_{n-p_n}|\mathcal{F}_{n-p_n})\mathbb{E}(S_{2n} - S_n|\mathcal{F}_n)|\mathcal{F}_0)\|_{3/2} < \infty . \tag{5.47}$$

We first notice that

$$\mathbb{E}(\mathbb{E}(S_{n-p_n}|\mathcal{F}_{n-p_n})\mathbb{E}(S_{2n} - S_n|\mathcal{F}_n)|\mathcal{F}_0) = \mathbb{E}(\mathbb{E}(S_{n-p_n}|\mathcal{F}_{n-p_n})\mathbb{E}(S_{2n} - S_n|\mathcal{F}_{n-p_n})|\mathcal{F}_0) ,$$

and by stationarity and (5.40)

$$\begin{aligned}\|\mathbb{E}(\mathbb{E}(S_{n-p_n}|\mathcal{F}_{n-p_n})\mathbb{E}(S_{2n}-S_n|\mathcal{F}_{n-p_n})|\mathcal{F}_0)\|_{3/2} &\leq \|S_{n-p_n}\|_3\|\mathbb{E}(S_{2n}-S_n|\mathcal{F}_{n-p_n})\|_3\\ &\leq C\sqrt{n}\|\mathbb{E}(S_{n+p_n}-S_{p_n}|\mathcal{F}_0)\|_3\,.\end{aligned}$$

Hence (5.47) will hold provided that

$$\sum_{n\geq 1}\frac{1}{n}\Big\|\sum_{k\geq[\sqrt{n}]}\mathbb{E}(X_k|\mathcal{F}_0)\Big\|_3<\infty\,. \tag{5.48}$$

The fact that (5.48) holds under the first part of the condition (3.4) follows from the following elementary lemma applied to $h(x)=\|\sum_{k\geq[x]}\mathbb{E}(X_k|\mathcal{F}_0)\|_3$.

Lemma 5.3. *Assume that h is a positive function on $\mathbb{R}^+$ satisfying $h(\sqrt{x+1})=h(\sqrt{n})$ for any x in $[n-1,n[$. Then $\sum_{n\geq 1}n^{-1}h(\sqrt{n})<\infty$ if and only if $\sum_{n\geq 1}n^{-1}h(n)<\infty$.*

It remains to show that

$$\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}\|\mathbb{E}(S_n\tilde{R}_n|\mathcal{F}_0)\|_{3/2}<\infty\,. \tag{5.49}$$

Write

$$\begin{aligned}S_n\tilde{R}_n &= S_n\Big(\sum_{k\geq 0}(X_{-k}-\mathbb{E}(X_{-k}|\mathcal{F}_0))-\sum_{k\geq -n}(X_{-k}-\mathbb{E}(X_{-k}|\mathcal{F}_n))\Big)\\ &= S_n\Big(\mathbb{E}(S_n|\mathcal{F}_n)-S_n+\sum_{k\geq 0}(\mathbb{E}(X_{-k}|\mathcal{F}_n)-\mathbb{E}(X_{-k}|\mathcal{F}_0))\Big)\,.\end{aligned}$$

Notice first that

$$\begin{aligned}\|\mathbb{E}\big(S_n(S_n-\mathbb{E}(S_n|\mathcal{F}_n))|\mathcal{F}_0\big)\|_{3/2} &= \|\mathbb{E}\big((S_n-\mathbb{E}(S_n|\mathcal{F}_n))^2|\mathcal{F}_0\big)\|_{3/2}\\ &\leq \|S_n-\mathbb{E}(S_n|\mathcal{F}_n)\|_3^2\,,\end{aligned}$$

which is bounded under the second part of the condition (3.4). Now for $p_n=[\sqrt{n}]$, we write

$$\sum_{k\geq 0}(\mathbb{E}(X_{-k}|\mathcal{F}_n)-\mathbb{E}(X_{-k}|\mathcal{F}_0))=\sum_{k\geq 0}(\mathbb{E}(X_{-k}|\mathcal{F}_n)-\mathbb{E}(X_{-k}|\mathcal{F}_{p_n}))+\sum_{k\geq 0}(\mathbb{E}(X_{-k}|\mathcal{F}_{p_n})-\mathbb{E}(X_{-k}|\mathcal{F}_0))\,.$$

Note that

$$\begin{aligned}\Big\|\sum_{k\geq 0}(\mathbb{E}(X_{-k}|\mathcal{F}_{p_n})-\mathbb{E}(X_{-k}|\mathcal{F}_0))\Big\|_3 &= \Big\|\sum_{k\geq 0}(X_{-k}-\mathbb{E}(X_{-k}|\mathcal{F}_0))-\sum_{k\geq 0}(X_{-k}-(\mathbb{E}(X_{-k}|\mathcal{F}_{p_n}))\Big\|_3\\ &\leq \Big\|\sum_{k\geq 0}(X_{-k}-\mathbb{E}(X_{-k}|\mathcal{F}_0))\Big\|_3+\Big\|\sum_{k\geq p_n}(X_{-k}-(\mathbb{E}(X_{-k}|\mathcal{F}_0))\Big\|_3\,,\end{aligned}$$

which is bounded under the second part of the condition (3.4). Next, since the random variable $\sum_{k\geq 0}(\mathbb{E}(X_{-k}|\mathcal{F}_{p_n}) - \mathbb{E}(X_{-k}|\mathcal{F}_0))$ is $\mathcal{F}_{p_n}$-measurable, we get

$$\begin{aligned}
&\Big\|\mathbb{E}\Big(S_n \sum_{k\geq 0}(\mathbb{E}(X_{-k}|\mathcal{F}_{p_n}) - \mathbb{E}(X_{-k}|\mathcal{F}_0))|\mathcal{F}_0\Big)\Big\|_{3/2} \\
&\leq \Big\|\mathbb{E}\Big(S_{p_n} \sum_{k\geq 0}(\mathbb{E}(X_{-k}|\mathcal{F}_{p_n}) - \mathbb{E}(X_{-k}|\mathcal{F}_0))|\mathcal{F}_0\Big)\Big\|_{3/2} \\
&\qquad + \|\mathbb{E}(S_n - S_{p_n}|\mathcal{F}_{p_n})\|_3 \Big\| \sum_{k\geq 0}(\mathbb{E}(X_{-k}|\mathcal{F}_{p_n}) - \mathbb{E}(X_{-k}|\mathcal{F}_0))\Big\|_3 \\
&\leq \Big(\|S_{p_n}\|_3 + \|\mathbb{E}(S_{n-p_n}|\mathcal{F}_0)\|_3\Big) \Big\| \sum_{k\geq 0}(\mathbb{E}(X_{-k}|\mathcal{F}_{p_n}) - \mathbb{E}(X_{-k}|\mathcal{F}_0))\Big\|_3 \leq C\sqrt{p_n}\,,
\end{aligned}$$

by using (3.4) and (5.40). Hence, since $p_n = [\sqrt{n}]$, we get that

$$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \Big\|\mathbb{E}\Big(S_n \sum_{k\geq 0}(\mathbb{E}(X_{-k}|\mathcal{F}_{p_n}) - \mathbb{E}(X_{-k}|\mathcal{F}_0))\Big|\mathcal{F}_0\Big)\Big\|_{3/2} < \infty\,.$$

It remains to show that

$$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \Big\|\mathbb{E}\Big(S_n \sum_{k\geq 0}(\mathbb{E}(X_{-k}|\mathcal{F}_n) - \mathbb{E}(X_{-k}|\mathcal{F}_{p_n}))\Big|\mathcal{F}_0\Big)\Big\|_{3/2} < \infty\,. \tag{5.50}$$

Note first that

$$\begin{aligned}
\Big\| \sum_{k\geq 0}(\mathbb{E}(X_{-k}|\mathcal{F}_n) - \mathbb{E}(X_{-k}|\mathcal{F}_{p_n}))\Big\|_3 &= \Big\| \sum_{k\geq 0}(X_{-k} - \mathbb{E}(X_{-k}|\mathcal{F}_n)) - \sum_{k\geq 0}(X_{-k} - (\mathbb{E}(X_{-k}|\mathcal{F}_{p_n}))\Big\|_3 \\
&\leq \Big\| \sum_{k\geq n}(X_{-k} - \mathbb{E}(X_{-k}|\mathcal{F}_0))\|_3 + \| \sum_{k\geq p_n}(X_{-k} - (\mathbb{E}(X_{-k}|\mathcal{F}_0))\Big\|_3\,.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\Big\|\mathbb{E}\Big(S_n \sum_{k\geq 0}(\mathbb{E}(X_{-k}|\mathcal{F}_n) - \mathbb{E}(X_{-k}|\mathcal{F}_{p_n}))|\mathcal{F}_0\Big)\Big\|_{3/2} \\
&\qquad \leq C\sqrt{n}\Big(\Big| \sum_{k\geq p_n}(X_{-k} - (\mathbb{E}(X_{-k}|\mathcal{F}_0)) + \Big\|_3 + \Big\| \sum_{k\geq n}(X_{-k} - (\mathbb{E}(X_{-k}|\mathcal{F}_0))\Big\|_3\Big)\,.
\end{aligned}$$

by taking into account (5.40). Consequently (5.50) will follow as soon as

$$\sum_{n\geq 1} \frac{1}{n} \Big\| \sum_{k\geq [\sqrt{n}]}(X_{-k} - \mathbb{E}(X_{-k}|\mathcal{F}_0))\Big\|_3 < \infty\,,$$

which holds under the second part of the condition (3.4), by applying Lemma 5.3 with $h(x) = \| \sum_{k\geq [x]}(X_{-k} - \mathbb{E}(X_{-k}|\mathcal{F}_0))\|_3$. This ends the proof of the theorem.

6 Appendix

6.1 A smoothing lemma.

Lemma 6.1. *Let $r > 0$ and f be a function such that $|f|_{\Lambda_r} < \infty$ (see Notation 5.1 for the definition of the seminorm $|\cdot|_{\Lambda_r}$). Let ϕ_t be the density of the law $N(0,t^2)$. For any real $p \geq r$ and any positive t, $|f * \phi_t|_{\Lambda_p} \leq c_{r,p} t^{r-p} |f|_{\Lambda_r}$ for some positive constant $c_{r,p}$ depending only on r and p. Furthermore $c_{r,r} = 1$.*

Remark 6.1. In the case where p is a positive integer, the result of Lemma 6.1 can be written as $\|f * \phi_t^{(p)}\|_\infty \leq c_{r,p} t^{r-p} |f|_{\Lambda_r}$.

Proof of Lemma 6.1. Let j be the integer such that $j < r \leq j+1$. In the case where p is a positive integer, we have

$$(f * \phi_t)^{(p)}(x) = \int \big(f^{(j)}(u) - f^{(j)}(x)\big) \phi_t^{(p-j)}(x-u) du \qquad \text{since } p - j \geq 1 \,.$$

Since $|f^{(j)}(u) - f^{(j)}(x)| \leq |x-u|^{r-j} |f|_{\Lambda_r}$, we obtain that

$$|(f * \phi_t)^{(p)}(x)| \leq |f|_{\Lambda_r} \int |x-u|^{r-j} |\phi_t^{(p-j)}(x-u)| du \leq |f|_{\Lambda_r} \int |u|^{r-j} |\phi_t^{(p-j)}(u)| du \,.$$

Using that $\phi_t^{(p-j)}(x) = t^{-p+j-1} \phi_1^{(p-j)}(x/t)$, we conclude that Lemma 6.1 holds with the constant $c_{r,p} = \int |z|^{r-j} \phi_1^{p-j}(z) dz$.

The case $p = r$ is straightforward. In the case where p is such that $j < r < p < j+1$, by definition

$$|f^{(j)} * \phi_t(x) - f^{(j)} * \phi_t(y)| \leq |f|_{\Lambda_r} |x-y|^{r-j} \,.$$

Also, by Lemma 6.1 applied with $p = j+1$,

$$|f^{(j)} * \phi_t(x) - f^{(j)} * \phi_t(y)| \leq |x-y| \|f^{(j+1)} * \phi_t\|_\infty \leq |f|_{\Lambda_r} c_{r,j+1} t^{r-j-1} |x-y| \,.$$

Hence by interpolation,

$$|f^{(j)} * \phi_t(x) - f^{(j)} * \phi_t(y)| \leq |f|_{\Lambda_r} t^{r-p} c_{r,j+1}^{(p-r)/(j+1-r)} |x-y|^{p-j} \,.$$

It remains to consider the case where $r \leq i < p \leq i+1$. By Lemma 6.1 applied successively with $p = i$ and $p = i+1$, we obtain that

$$|f^{(i+1)} * \phi_t(x)| \leq |f|_{\Lambda_r} c_{r,i+1} t^{r-i-1} \quad \text{and} \quad |f^{(i)} * \phi_t(x)| \leq |f|_{\Lambda_r} c_{r,i} t^{r-i} \,.$$

Consequently

$$|f^{(i)} * \phi_t(x) - f^{(i)} * \phi_t(y)| \leq |f|_{\Lambda_r} t^{r-i} (2c_{r,i} \wedge c_{r,i+1} t^{-1} |x-y|) \,,$$

and by interpolation,

$$|f^{(i)} * \phi_t(x) - f^{(i)} * \phi_t(y)| \leq |f|_{\Lambda_r} t^{r-p} (2c_{r,i})^{1-p+i} c_{r,i+1}^{p-i} |x-y|^{p-i} \,.$$

6.2 Covariance inequalities.

In this section, we give an upper bound for the expectation of the product of k centered random variables $\Pi_{i=1}^k(X_i - \mathbb{E}(X_i))$.

Proposition 6.1. *Let $X = (X_1, \cdots, X_k)$ be a random variable with values in $\mathbb{R}^k$. Define the number*

$$\begin{aligned}\phi^{(i)} &= \phi(\sigma(X_i), X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_k) \qquad (6.1)\\ &= \sup_{x\in\mathbb{R}^k} \Big\| \mathbb{E}\Big(\prod_{j=1, j\neq i}^k (\mathbb{1}_{X_j > x_i} - \mathbb{P}(X_i > x_i)) | \sigma(X_i)\Big) - \mathbb{E}\Big(\prod_{j=1, j\neq i}^k (\mathbb{1}_{X_j > x_i} - \mathbb{P}(X_i > x_i))\Big)\Big\|_\infty .\end{aligned}$$

Let F_i be the distribution function of X_i and Q_i be the quantile function of $|X_i|$ (see Section 4.1 for the definition). Let F_i^{-1} be the generalized inverse of F_i and let $D_i(u) = (F_i^{-1}(1-u) - F_i^{-1}(u))_+$. We have the inequalities

$$\Big|\mathbb{E}\Big(\prod_{i=1}^k X_i - \mathbb{E}(X_i)\Big)\Big| \leq \int_0^1 \Big(\prod_{i=1}^k D_i(u/\phi^{(i)})\Big) du \qquad (6.2)$$

and

$$\Big|\mathbb{E}\Big(\prod_{i=1}^k X_i - \mathbb{E}(X_i)\Big)\Big| \leq 2^k \int_0^1 \Big(\prod_{i=1}^k Q_i(u/\phi^{(i)})\Big) du\,. \qquad (6.3)$$

In addition, for any k-tuple $(p_1, \ldots, p_k)$ such that $1/p_1 + \ldots + 1/p_k = 1$, we have

$$\Big|\mathbb{E}\Big(\prod_{i=1}^k X_i - \mathbb{E}(X_i)\Big)\Big| \leq 2^k \prod_{i=1}^k (\phi^{(i)})^{1/p_i} \|X_i\|_{p_i}\,. \qquad (6.4)$$

Proof of Proposition 6.1. We have that

$$\mathbb{E}\Big(\prod_{i=1}^k X_i - \mathbb{E}(X_i)\Big) = \int \mathbb{E}\Big(\prod_{i=1}^k \mathbb{1}_{X_i > x_i} - \mathbb{P}(X_i > x_i)\Big) dx_1 \ldots dx_k\,. \qquad (6.5)$$

Now for all i,

$$\begin{aligned}&\mathbb{E}\Big(\prod_{i=1}^k \mathbb{1}_{X_i > x_i} - \mathbb{P}(X_i > x_i)\Big)\\ &= \mathbb{E}\left(\mathbb{1}_{X_i > x_i}\Big(\mathbb{E}\Big(\prod_{j=1, j\neq i}^k (\mathbb{1}_{X_j > x_i} - \mathbb{P}(X_i > x_i)) | \sigma(X_i)\Big) - \mathbb{E}\Big(\prod_{j=1, j\neq i}^k (\mathbb{1}_{X_j > x_i} - \mathbb{P}(X_i > x_i))\Big)\Big)\right)\\ &= \mathbb{E}\left(\mathbb{1}_{X_i \leq x_i}\Big(\mathbb{E}\Big(\prod_{j=1, j\neq i}^k (\mathbb{1}_{X_j > x_i} - \mathbb{P}(X_i > x_i)) | \sigma(X_i)\Big) - \mathbb{E}\Big(\prod_{j=1, j\neq i}^k (\mathbb{1}_{X_j > x_i} - \mathbb{P}(X_i > x_i))\Big)\Big)\right).\end{aligned}$$

Consequently, for all i,

$$\mathbb{E}\Big(\prod_{i=1}^{k} \mathbb{1}_{X_i>x_i} - \mathbb{P}(X_i > x_i)\Big) \le \phi^{(i)}\mathbb{P}(X_i \le x_i) \wedge \mathbb{P}(X_i > x_i)\,. \tag{6.6}$$

Hence, we obtain from (6.5) and (6.6) that

$$\begin{aligned}\Big|\mathbb{E}\Big(\prod_{i=1}^{k} X_i - \mathbb{E}(X_i)\Big)\Big| &\le \int_0^1 \Big(\prod_{i=1}^{k}\int \mathbb{1}_{u/\phi^{(i)}<\mathbb{P}(X_i>x_i)}\mathbb{1}_{u/\phi^{(i)}\le\mathbb{P}(X_i\le x_i)}dx_i\Big)du \\ &\le \int_0^1 \Big(\prod_{i=1}^{k}\int \mathbb{1}_{F_i^{-1}(u/\phi^{(i)})\le x_i<F_i^{-1}(1-u/\phi^{(i)})}dx_i\Big)du,\end{aligned}$$

and (6.2) follows. Now (6.3) comes from (6.2) and the fact that $D_i(u) \le 2Q_i(u)$ (see Lemma 6.1 in Dedecker and Rio (2006)). □

Definition 6.1. For a quantile function Q in $\mathbb{L}_1([0,1],\lambda)$, let $\mathcal{F}(Q,P_X)$ be the set of functions f which are nondecreasing on some open interval of $\mathbb{R}$ and null elsewhere and such that $Q_{|f(X)|} \le Q$. Let $\mathcal{C}(Q,P_X)$ denote the set of convex combinations $\sum_{i=1}^{\infty}\lambda_i f_i$ of functions f_i in $\mathcal{F}(Q,P_X)$ where $\sum_{i=1}^{\infty}|\lambda_i| \le 1$ (note that the series $\sum_{i=1}^{\infty}\lambda_i f_i(X)$ converges almost surely and in $\mathbb{L}_1(P_X)$).

Corollary 6.1. *Let $X=(X_1,\cdots,X_k)$ be a random variable with values in $\mathbb{R}^k$ and let the $\phi^{(i)}$'s be defined by (6.1). Let $(f_i)_{1\le i\le k}$ be k functions from $\mathbb{R}$ to $\mathbb{R}$, such that $f_i \in \mathcal{C}(Q_i,P_{X_i})$. We have the inequality*

$$\Big|\mathbb{E}\Big(\prod_{i=1}^{k} f_i(X_i) - \mathbb{E}(f_i(X_i))\Big)\Big| \le 2^{2k-1}\int_0^1 \prod_{i=1}^{k} Q_i\Big(\frac{u}{\phi^{(i)}}\Big)du\,.$$

Proof of Corollary 6.1. Write for all $1\le i\le k$, $f_i = \sum_{j=1}^{\infty}\lambda_{j,i}f_{j,i}$ where $\sum_{j=1}^{\infty}|\lambda_{j,i}| \le 1$ and $f_{j,i} \in \mathcal{F}(Q_i,P_{X_i})$. Clearly

$$\begin{aligned}\Big|\mathbb{E}\Big(\prod_{i=1}^{k} f_i(X_i) - \mathbb{E}(f_i(X_i))\Big)\Big| &\le \sum_{j_1=1}^{\infty}\cdots\sum_{j_k=1}^{\infty}\Big(\prod_{i=1}^{k}|\lambda_{j_i,i}|\Big)\Big|\mathbb{E}\Big(\prod_{i=1}^{k} f_{j_i,i}(X_i) - \mathbb{E}(f_{j_i,i}(X_i))\Big)\Big| \\ &\le \sup_{j_1\ge1,\ldots,j_k\ge1}\Big|\mathbb{E}\Big(\prod_{i=1}^{k} f_{j_i,i}(X_i) - \mathbb{E}(f_{j_i,i}(X_i))\Big)\Big|\,. \end{aligned}\tag{6.7}$$

Since each $f_{j_i,i}$ is nondecreasing on some interval,

$$\phi(\sigma(f_{j_i,i}(X_i)), f_{j_1,1}(X_1),\ldots,f_{j_{i-1},i-1}(X_{i-1}), f_{j_{i+1},i+1}(X_{i+1}),\ldots,f_{j_k,k}(X_k)) \le 2^{k-1}\phi^{(i)}\,.$$

Then applying (6.3) on the right hand side of (6.7), we derive that

$$\Big|\mathbb{E}\Big(\prod_{i=1}^{k} f_i(X_i) - \mathbb{E}(f_i(X_i))\Big)\Big| \le 2^k\int_0^1 \prod_{i=1}^{k} Q_i\Big(\frac{u}{2^{k-1}\phi^{(i)}}\Big)du\,,$$

and the result follows by a change-of-variables. □

Recall that for any $p \geq 1$, the class $\mathcal{C}(p, M, P_X)$ has been introduced in the definition 4.2.

Corollary 6.2. *Let $X = (X_1, \cdots, X_k)$ be a random variable with values in $\mathbb{R}^k$ and let the $\phi^{(i)}$'s be defined by (6.1). Let a k-tuple $(p_1, \ldots, p_k)$ such that $1/p_1 + \ldots + 1/p_k = 1$ and let $(f_i)_{1 \leq i \leq k}$ be k functions from $\mathbb{R}$ to $\mathbb{R}$, such that $f_i \in \mathcal{C}(p_i, M_i, P_{X_i})$. We have the inequality*

$$\Big| \mathbb{E}\Big(\prod_{i=1}^{k} f_i(X_i) - \mathbb{E}(f_i(X_i)) \Big) \Big| \leq 2^{2k-1} \prod_{i=1}^{k} (\phi^{(i)})^{1/p_i} M_i^{1/p_i} \, .$$

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